INVESTIGATION OF CERTAIN DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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By

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CERTIFICATE

It is certified that the work contained in the thesis entitled **"Investigation of Certain Differential and Integro-differential Equations in Abstract Spaces"** by Ms. **Reeta Shukla (Roll No: 9920861)** has been carried out under my supervision. In my opinion, the thesis has reached the standard fulfilling the requirement of regulation of the Ph.D. degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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Synopsis

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Many physical problems can be modelled as evolution equations in Hilbert or more generally, in Banach spaces. The evolution equations may be viewed as ordinary differential equations in infinite dimensional spaces of functions and are generally associated with the partial differential equations modelling many physical phenomena such as the heat conduction in materials, the wave propagation in materials, the population dynamics, chemical reactions, etc. The work presented in this thesis aims to demonstrate the applicability of rich theory of functional analysis to analyze the existence, uniqueness, continuous dependence on the initial data and asymptotic behavior of the solutions of such problems.

The main advantage in analyzing such problems in abstract spaces is that not only we establish the results for the problems under consideration, but also for the whole class of problems to which these prototype of problems belong. We concentrate on basic features those remain invariant when we switch from one problem to another belonging

to the same class.

After analyzing the existence, uniqueness and continuous dependence of a solution of an evolution equation, we aim to find the solution. Often it is a difficult task to find the exact solution. In such a case, the approximate methods of analysis provide alternate ways of finding the approximate solutions and analyzing their behaviors. We consider certain approximate equations in finite dimensional spaces associated with an evolution equation under consideration and establish the existence and uniqueness of solutions to these approximate equations. The convergence of the solutions of the approximate equations to the solution of the evolution equation is then established. Our tools for this are the theory of analytic semigroups and the contraction mapping theorem. We consider analytic semigroups generated by the operators appearing in evolution equations under consideration.

Although, our study concerns some special type of evolution equations, namely the Sobolev type equations and certain classes of second order semi-linear differential and semi-linear integro-differential equations but the techniques are quite general extendable to other type of problems. The results obtained may be applied to evolution equations arising in the modelling of many physical problems, for instance, the propagation of waves with strong damping, the vibrations in a beam, certain viscoelastic materials with memory, the flow of a fluid through fissured rocks and the shear in a second-order fluid and other related problems.

The thesis has seven chapters.

In **Chapter 1**, first we give the introduction to the problems discussed in the subsequent chapters providing a motivation to the study carried out in this thesis. Then, we give a review of related works in the literature and the significance of the problems considered. Next, we mention some of the results required for the analysis in

the ensuing chapters and finally, we give a brief description of the analysis carried out in the ensuing chapters.

In **Chapter 2**, we consider the Sobolev type evolution equations in a separable Hilbert space and study the approximations of the solutions. We extend the technique of Milleta (1994) to these type of problems. The Sobolev type evolution equations are implicit evolution equations in which time derivatives of the unknown function appear implicitly and these type of equations arise in many diffusion and fluid flow models. First of all, we consider preliminaries and assumptions required later. We then consider an associated integral equation and a sequence of approximate integral equations using projection operators. We establish the existence of a unique solution to every approximate integral equation using the contraction mapping theorem. After proving some estimates for the solutions of approximate integral equations, we prove the convergence of the solutions of the approximate integral equations to the solution of the associated integral equation. Further, we show that the solution of associated integral equation can be extended to the maximal interval of existence and it is unique. Finally, we consider the Faedo-Galerkin approximations of solutions and prove some convergence results.

In **Chapter 3**, we consider a strongly damped semi-linear wave equation and reformulate it as a second order semi-linear evolution equation. We study the approximation of the solution to a second order semi-linear evolution equation in a separable Hilbert space by using similar techniques which are used in Chapter 2. First, with the help of a pair of associated integral equations and projection operators, we consider a pair of approximate integral equations and prove the existence and uniqueness of the solution to this pair. We then establish the convergence of the pair of approximate integral equations to the pair of associated integral equations and limit of the solution of approximate integral equations being the solution of associated integral equation

after proving some estimates for the solutions of the approximate integral equations. Further, we show that the solution of the pair of associated integral equations can be extended to the maximal interval of existence and it is unique. Finally, we consider the Faedo-Galerkin approximations of the solutions and prove some convergence results.

The behavior of many physical systems at an instant depends not only on the instantaneous values of the functions appearing in the governing equations but on their values at previous instants also. Such a system is called a *system with a memory.* For instance, in nuclear dynamics and thermo-elasticity, there is a need to reflect the effects of the "memory" of the system. This results in the inclusion of an integral term into the governing partial differential equation yielding a partial integro-differential equation (PIDE). This provides us a motivation to study the partial integro-differential equations. In the next two Chapters, we study a class of second order semi-linear integro-differential equations arising in the study of viscoelastic material with memory.

In **Chapter 4**, we consider an abstract second order semi-linear integro-differential equation in a Banach space and first prove the existence and uniqueness of a local classical solution with the help of the semigroup theory and the contraction mapping theorem. Further, under some additional growth conditions on the nonlinear maps we establish the continuation of this solution, the maximum interval of existence and the global existence.

In **Chapter 5**, we consider a strongly damped semi-linear integro-differential equation and reformulate it as a second order semi-linear integro-differential equation in a separable Hilbert space and study the convergence of the approximation of the solution using similar techniques which are used in Chapters 2 and 3.

In **Chapter 6**, we consider a first order quasi-linear implicit integro-differential equation in a Banach space. First, we prove the existence, uniqueness and continu-

ous dependence on the initial data of a strong solution by using the method of semi-discretization in time in which we discretize the time axis and replace the time derivatives by the correspond difference quotients. Then, we show that these discrete points lies in a ball whose radius is independent of the discretization parameters and prove some estimates. After defining the approximate solution in terms of these discrete points, we prove its convergence. Next, we establish the existence of a unique local mild solution and its regularity under some additional condition with the help of the semigroup theory and the contraction mapping theorem.

In **Chapter 7**, we conclude the thesis with some remarks and provide some insight into further research work in this and related areas.

Relevant references are appended at the end.

Dedicated

To

My Parents

Late Mrs. Suresh Kumari Shukla

and

Mr. Kaushal Kishore Shukla

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Fantasy, energy, self-confidence

and self-criticism are the characteristic endowments

of the mathematician.

— Sophus Lie

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Chapter 1

Introduction and Preliminaries

In so far as the theorems of the mathematics relate to the reality they are not certain, and in so far as they are certain they do not relate to reality.

— Albert Einstein

1.1 Introduction

Many physical problems can be modelled as evolution equations in Hilbert spaces or more generally, in Banach spaces. In this dissertation we have studied some classes of evolution equations which arise in several physical and realistic phenomena, for example, the propagation of waves with strong damping, the vibrations in a beam, the viscoelastic materials with memory, the flow of a fluid through fissured rocks and the shear in a second-order fluid. We have established some results for the existence, uniqueness and convergence of the approximation of solutions to certain nonlinear first and second order evolution equations. By the order of an evolution equation we mean the highest order of the time derivative appearing in the evolution equation.

After proving the existence of a unique solution to an evolution equation in an abstract space, the next aim is to find the solution. Often it is a difficult task to find the exact solution. In the such cases, the approximation methods of analysis provide alternate ways of finding the approximate solutions and analyzing their behaviors. We consider certain approximate equations in finite dimensional spaces associated with the evolution equations under consideration and establish the existence and uniqueness of the solutions to these approximate equations. The convergence of the solutions of the approximate equations to the solution of the evolution equation is then established. Similar procedure is followed to study the nonlinear Sobolev type evolution equations and the second order semi-linear evolution equations. Further, we prove the convergence of the Faedo-Galerkin approximations to the solutions whose existence and uniqueness have already been established. The Sobolev type evolution equations arise in the study of the partial neutral functional differential equations with unbounded delay. The strongly damped semi-linear wave equation, the Klein-Gordon equation, the dynamical von-Karman equation and the vibrating beam equation can be modelled as the second order semi-linear evolution equations. Next, we have considered the existence and uniqueness of the solutions to second order semi-linear integro-differential equations which arise in the study of the viscoelastic materials with memory and have proved the convergence of the approximate solutions. Finally, we have considered the first order quasi-linear implicit integro-differential equations and established some existence and uniqueness results.

Certain partial differential equations are of widespread interest because of their connection with phenomena in the physical world. The heat conduction in materials, the vibrations in the wires and beams, the motion of the elastic bodies, the propagation of a small disturbance in a gas and many physical problems are studied by using their governing partial differential equations. These partial differential equations are classified according to certain common features prevailing in each class. We concentrate on

certain prototype of problems, which represent each of the members of these classes. For instance, the heat equation, the wave equation and the Laplace equation are the prototype of the parabolic, hyperbolic and elliptic equations, respectively. Many properties remain same for each member of the same class. These properties are called the *invariant properties* of the class.

If we are interested in the invariant properties of certain class of problems then the best way to study such problems is to consider the abstract formulations of such problems. By an abstract formulation we mean a functional analytic representation of the problems. The abstract formulations of the prototype of problems are known as the evolution equations and their study comes under the area which has the AMS classification as *Differential Equation in Abstract Spaces (34Gxx).* These are basically ordinary differential equations in the Hilbert spaces or more generally in the Banach spaces of functions. In such formulations only invariant properties come into picture and many unnecessary details of an individual problem get suppressed. This allows us to concentrate on the invariant properties of a class. We may then use the rich theory of the functional analysis to establish the existence, uniqueness, continuous dependence on the initial data and asymptotic behavior of the solutions.

To successfully exploit the tools of the functional analysis, one has to choose a suitable space and properly define the operators in that space. Same problem may be reformulated differently in different spaces. Therefore it is very important to make a proper choice of the space and the operators required for the abstract formulations. The main advantage in analyzing such problems in the abstract spaces is that not only we establish the results for the problem under consideration, but for the whole class of problems also to which the prototype of problem belongs. For example, the results established for the abstract formulation of the strongly damped semi-linear wave equation can be applied to the Klein-Gordon Equation, the dynamical von-Karman

equation and the vibrating beam equation. We concentrate only on the basic features those remain invariant when we switch from one problem to another belonging to the same class.

The evolution of a physical system in time is usually described by an initial value problem for a differential equation. The differential equation may be an ordinary or a partial and mixed initial-boundary value problems are included. Let $u(t)$ describe the state of some physical system at the time t. Suppose that the time rate of change of $u(t)$ is given by some function A of the state of the system $u(t)$ and the initial data $u(0) = f$ is also given. Then we have the initial value problem

$$\frac{du}{dt}(t) = Au(t), \quad t \geq 0, \quad u(0) = f. \tag{1.1.1}$$

The function u takes value in a Banach space X and A be an operator from its domain $D(A)$ in X to X.

Abstract Formulation

Now, we give a brief idea of converting a partial differential equation to an abstract form where the space variable is suppressed and the equation looks like an ordinary differential equation in time variable in an infinite dimensional space.

Example 1.1.1 Consider the Laplacian $\triangle = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ in a bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary $\partial\Omega$. Consider the initial boundary value problem for the heat equation

$$\begin{aligned}
&\frac{\partial w}{\partial t} = \triangle w, \quad t, x \in (0, \infty) \times \Omega, \\
&w(0, x) = f(x), \quad x \in \Omega, \\
&w(t, x) = 0, t, \quad x \in [0, \infty) \times \partial\Omega.
\end{aligned} \tag{1.1.2}$$

We set

$$u(t) = w(t, .),$$

regarded as a function of x and take X to be the space of functions on Ω e.g., $L^p(\Omega)$ for $p \geq 1$ or $C(\bar{\Omega})$. The derivatives

$$\frac{du}{dt} \quad \text{and} \quad \frac{\partial w}{\partial t}$$

are both limits of the difference quotient

$$\frac{w(t+h, x) - w(t, x)}{h},$$

first limit being in the sense of the norm of X and the second limit being a pointwise one. We may formally identify $\frac{\partial w}{\partial t}$ with $\frac{du}{dt}$.

To define A we take $X = L^2(\Omega)$ and let $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Define

$$Av = \triangle v \quad \text{for} \quad v \in D(A).$$

Equation (1.1.2) are thus written in the abstract form (1.1.1).

Remark 1.1.2 The boundary conditions of (1.1.2) is absorbed into the domain of definition of operator A and into the requirement that $u(t) \in D(A)$ for all $t \geq 0$.

Example 1.1.3 Consider the initial value problem for the wave equation in $\mathbf{R}^N$

$$\begin{aligned}
&\frac{\partial^2 w}{\partial t^2} = \triangle w, \quad x \in \mathbf{R}^N, t > 0, \\
&w(0, x) = f_1(x), \quad x \in \mathbf{R}^N, \\
&\frac{\partial w}{\partial t}(0, x) = f_2(x), \quad x \in \mathbf{R}^N.
\end{aligned} \tag{1.1.3}$$

Let $X = H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ and the domain of A, $D(A) = H^2(\mathbf{R}^N) \times H^1(\mathbf{R}^N)$. We set

$$u(t) = \begin{pmatrix} w(t, .) \\ \frac{\partial w}{\partial t}(t, .) \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix},$$

i.e.

$$A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ \triangle v_1 \end{pmatrix}.$$

Then the equation (1.1.3) is converted into (1.1.1).

Thus, we see that the evolution equations in some sense are the initial value problems for the ordinary differential equations in the infinite dimensional spaces and are associated with the partial differential equations governing certain physical phenomena.

In this dissertation we have considered the following type of evolution equations.

- First Order Evolution Equations.
  - (a) Implicit Evolution Equations.
  - (b) Quasi-linear integro-differential Equations.
- Second Order Evolution Equations.
  - (a) semi-linear Evolution Equations.
  - (b) semi-linear integro-differential Equations.

The Sobolev type evolution equations are implicit evolution equations in which the time derivatives of the unknown function are not given explicitly. These type of equations arise in the study of various diffusion and fluid flow models. Consider an initial boundary value problem on a cylinder of the form

$$D_t(u + b(x)Lu) + Lu = f(t, x, u), \tag{1.1.4}$$

where Lu is a nonlinear elliptic operator in the divergence form. The coefficient $b(.)$ is assumed to be bounded, measurable and nonnegative.The above equation is parabolic

where $b(x) = 0$ and of the Sobolev type where $b(x) > 0$. The coefficient $b(.)$ represents a quantity with the dimensions of the viscosity. Applications of these and related results are considered in [86]. We may write (1.1.4) in an abstract form by choosing a proper space and an operator as

$$\frac{d}{dt}(u(t) + g(t, u(t))) + Au(t) = f(t, u(t)). \tag{1.1.5}$$

A partial neutral functional differential equation with unbounded delay is a particular case of the nonlinear Sobolev type evolution equation of the form

$$\frac{d}{dt}(u(t) + G(t, u_t)) = Au(t) + F(t, u_t), \quad t > 0, \tag{1.1.6}$$

in a Banach space X where A is the infinitesimal generator of an analytic semigroup in X, F and G are appropriate nonlinear functions from $[0, T] \times W$ into X and for any function $u \in C((-\infty, \infty), X)$, the history function $u_t \in C((-\infty, 0], X)$ of u is given by $u_t(\theta) = u(t+\theta)$. The above equation is called the *abstract neutral functional differential equation* (ANFDE) with unbounded delay.

As a motivational example for this class of equations we consider the boundary value problem

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,x) + \int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,\eta,x)u(s,\eta)d\eta ds\right] = \frac{\partial^2}{\partial x^2}u(t,x) + a_0(x)u(t,x),\\
&\qquad + \int_{-\infty}^{t} a(s-t)u(s,x)ds + a_1(t,x), \quad t \geq 0, \quad 0 \leq x \leq \pi,\\
&u(t,0) = u(t,\pi) = 0, \quad t \geq 0,\\
&u(\theta,x) = \phi(\theta,x), \quad \theta \leq 0, \quad 0 \leq x \leq \pi,
\end{aligned}$$

where the functions a_0, a, a_1, b_1 and ϕ satisfy appropriate conditions.

The partial integro-differential equations with infinite delays arising, for example, in the study of heat conduction in the materials with memory or population dynamics for spatially distributed population, may be described in the form

$$\frac{d}{dt}u(t) = Au(t) + F(t, u_t), \quad t \geq 0, \tag{1.1.7}$$

where A is the infinitesimal generator of a strongly continuous semigroup of linear operators on a Banach space X. The above equation is called an *abstract retarded functional differential equation* (ARFDE). Some ARFDEs can be conveniently transformed into ANFDEs. For example, consider equation (1.1.7) with

$$F(\varphi) = \int_{-\infty}^{0} C(-\theta)\varphi(\theta)d\theta,$$

where $C(s)$ is a strongly continuous map of continuous operator from X into X. Assuming that we can decompose $C(s) = L(s) + N(s)$ where $L(s)$ and $N(s)$ are also strongly continuous maps of continuous operators and further $L(s)$ is linear. We define the operator $V(t)$ by

$$V(t)u := \int_{0}^{t} L(s)uds.$$

Proceeding formally, the equation (1.1.7) may be transformed into an ANFDE as

$$\frac{d}{dt}\left[u(t) + \int_{-\infty}^{t} V(t-s)u(s)ds\right] = Au(t) + \int_{-\infty}^{t} N(t-s)u(s)ds,$$

which has the form (1.1.6) and in some cases, depending on V and N, it is easier to treat this equation than the original one.

The equation (1.1.5) with a linear g arises in, for example, in thermodynamics, in the flow of a fluid through fissured rocks, in the shear in second-order fluids and soil mechanics. Several applications of the results for the Sobolev type evolution equations to physical problems provide us a motivation to the establish the convergence results for the approximate solutions of these equations.

The behavior of many physical systems at an instant depends not only the instantaneous values of the functions appearing in the governing equations but their values at previous instants also. Such a system is called a *system with a memory.* For instance, in nuclear dynamics and thermoelasticity there is a need to reflect the effects of the "memory" of the system. This results in the inclusion of an integral term into the governing partial differential equation yielding a partial integro-differential equation (PIDE).

As an example, Consider the heat conduction equation for a one dimensional rod of unit length. In this model the temperature, $u(x,t)$, satisfies

$$D_t u(x,t) = D_{xx} u(x,t),$$

where D_t is the first partial derivative with respect to time and D_{xx} is the second derivative with respect to x, the space variable. In [76] Nunziato observes that the above system has two major shortcomings. First, this equation fails to take into account the memory effects present in some materials, particularly at low temperatures. Secondly this equation implies that a thermal disturbance at any point in the body will be felt instantaneously throughout the body. Several alternate formulations which include an integral term with respect to time have been considered in [18, 29, 71, 75, 76]. For example, [29] considers the following heat flow equation

$$D_t u(x,t) = \int_0^t h(t-s) D_x(\sigma(D_x u(x,s)) ds + k(x,t),$$

where the functions h, σ and k satisfy some appropriate conditions. This provides us a motivation to study the partial integro-differential equations. In this work we have studied certain classes of first and second order integro-differential equations.

In many practical applications, the coefficients of the partial differential operators depend also on the time and the unknown solution of the problem. Therefore associated operator in the abstract Cauchy problem involve time and the unknown function. We have considered a class of first order quasi-linear integro-differential equations.

For proving the existence and uniqueness results, we have used the theory of monotone operators, the theory of semigroups of operators and the method of semi-discretization in time which are very efficient tools in the study of nonlinear evolution equations. The method of semi-discretization in time is also know as the Rothe method or the method of lines. In this method using the time discretization, an evolution equation is approximated by the corresponding system of elliptic equations by means of which an approximate solution for the original evolution equation is constructed. After proving some *apriori* estimates for the approximate solution, the convergence of the approximate solution to the unique solution of the evolution problem is established. In these studies either global Lipschitz conditions or local Lipschitz conditions with some growth conditions on nonlinear forcing terms have been assumed.

The Rothe method has its significance both as a numerical method as well as a theoretical tool. The existence theorems are proved in a constructive way. Thus, no other methods are needed to give preliminary information on existence or regularity of the solutions as required in many other numerical methods when question on convergence or order of convergence etc. are to be answered. The Rothe method is a stable method. To the solution of elliptic problem generated by this method, the variational method can be applied. As the theoretical results are concerned, they are obtained in a relatively simple way. Moreover, the Rothe method, being a very natural one, makes it possible to get a particularly good insight into the structure of the solutions. Often a brief inspection of the corresponding elliptic problems gives an information what can be expected concerning the properties of the solution.

For the convergence results we are mainly concerned with the approximation of the solution to an evolution equation in a separable Hilbert space H by the Faedo-Galerkin method and our approach is that of numerical and analytical approximations. Let $\{u_i\}$ be orthonormal basis for Hilbert space. Then it is natural to consider the subspace

$H_n = span\{u_1, \cdots, u_n\}$ and the projection $P_n : H \to H_n$, defined by $P_n x = \sum_{i=1}^{n} x_i u_i$ for $x = \sum_{i=1}^{\infty} x_i u_i$. In our approach we consider the integral equations associated to the evolution equation. With the help of an associated integral equation and the projection operators we get an approximate integral equation which has a unique solution. Then we establish the convergence of a sequence of the approximate integral equations to the associated integral equation and show the limit of the solutions of the approximate integral equations be the solution of the associated integral equation. Then we consider the projections onto the finite dimensional subspaces H_n of the approximate integral equations. Finally, we consider the Faedo-Galerkin approximations and show that it converges to the solution of the evolution equation.

1.2 Literature Survey

If in an evolution equation, the coefficient of the highest order derivative cannot be so isolated, the equation is said to be of Sobolev type. For example the equation

$$(A(.)u)_{tt} + (B(.)u)_t + C(.)u = g$$

is of the Sobolev type. This type of problem called *singular* if at least one of the operator coefficient tends to infinity in some sense as $t \to 0$ and *degenerate* if some operator coefficient tends to zero as $t \to 0$ in such a way as to change the type of problem. In 1976 Carroll and Showalter has considered a subclass of singular and degenerate Sobolev equations in their book [26] and considered the well-posedness of these problems. They have considered the abstract Sobolev type equations

$$\frac{d}{dt}(M(u)) + L(u) = f, \tag{1.2.1}$$

$$\frac{d^2}{dt^2}(C(u)) + \frac{d}{dt}(B(u)) + A(u) = f \tag{1.2.2}$$

and have used the spectral and the energy methods to analyze (1.2.2) when C is the identity operator and B and A are (possibly degenerate) operator polynomials involving a closed densely defined self adjoint operator. The equation (1.2.1) is called *strongly regular* when M is invertible and $M^{-1}L$ is continuous. Similarly, (1.2.2) is strongly regular when $C^{-1}B$ and $C^{-1}A$ are continuous. The strongly regular case is studied in [26] and the operators are permitted to be time dependent and nonlinear. They also have given well-posedness results for the linear weakly regular equations in which only the leading operators are invertible, by means of the classical generation theory of linear semigroups in the Hilbert spaces.

Brill [25] has studied the abstract Cauchy problem for the semi-linear Sobolev type evolution equation

$$\frac{d}{dt}Bu(t) + Au(t) = g(u(t), t), \tag{1.2.3}$$

in which B and A are the linear operators with their domains contained in a real Banach space X and the ranges contained in a real Banach space Y. Function g maps $X \times [0, T]$, $T > 0$, into Y. He has developed an existence theory for (1.2.3) under the fundamental requirement that the operator B dominates the operator $A : D(B) \subset D(A)$. The abstract problem, he has considered arises as a realization of an initial boundary value problem for the pseudo-parabolic partial differential equation

$$\frac{\partial \mathcal{M}u(x,t)}{\partial t} + \mathcal{L}u(x,t) = \bar{g}(x, t, (D^{\alpha}u(x,t))_{|\alpha| \leq 2m-1}). \tag{1.2.4}$$

Here, $\mathcal{M}$ and $\mathcal{L}$ are the linear elliptic differential operators of orders $2m$ and $2l$, respectively, with $m \geq l$. The equation (1.2.4) appears in a variety of physical problems, for example, in the thermodynamics [27], in the flow of fluid through fissured rocks [19], in the shear in second-order fluids [94] and in the soil mechanics [93]. The case of a function $\bar{g}$ which satisfies a local Lipschitz condition has been treated by Showalter

[85, 86]. His results, which are proven without using compactness properties, apply only to a small class of nonlinearities.

Lightbourne and Rankin [67] have established an existence result for the Cauchy problem for the partial functional differential equation of the Sobolev type

$$(Bu(t))' + Au(t) = g(t, u_t), \quad t > 0, \tag{1.2.5}$$

$$u(t) = \phi(t), \quad -r \leq t \leq 0. \tag{1.2.6}$$

The function g is continuous and A and B are closed linear operators with their domains contained in a Banach space X and ranges in a Banach spaces Y. The approach taken in [67] is to consider the related integral equation

$$\begin{aligned} v(t) &= T(t)B\phi(0) + \int_0^t T(t-s)g(s, B^{-1}v_s)ds, \quad t \geq 0, \\ v(t) &= B\phi(t), \quad -r \leq t \leq 0, \end{aligned}$$

where $T(t)$, $t \geq 0$ is the semigroup of bounded linear operators generated by $-AB^{-1}$.

Hernández and Henŕiquez have established in [40] the existence of a mild and a strong solution while in [41] the result of the existence of a periodic solution for the quasi-linear partial neutral functional differential equation

$$\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t),$$

where A is the infinitesimal generator of a strongly continuous semigroup of the linear operators on a Banach space and F and G are the appropriate functions defined on a phase space.

In [39] Hernández have proved the existence of a regular solution for the quasi-linear evolution equation of the Sobolev type

$$\frac{d}{dt}(x(t) + g(t, x(t)) = Au(t) + f(t, x(t)),$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators defined on a Banach space and the functions f and g are continuous.

To solve an infinite dimensional equation, we adopt the most natural and elementary approach of replacing it by an appropriate finite dimensional problem. We then solve that approximate problem and prove the convergence to the solution of the original problem. For this, we consider a procedure by using the projections onto a finite dimensional space. This is referred as Galerkin approximations although every slight variant has its own name in literature e.g. 'Bubnov-Galerkin', 'Petrov-Galerkin', 'Faedo-Galerkin' etc. The Galerkin approximations used for the nonlinear equations are called the Faedo-Galerkin approximations.

Despite the widespread use of the Faedo-Galerkin method (in many applications it is referred to as the method of harmonic balance), the convergence behavior in many cases is not known. Bazley [20] has considered the IVP for a nonlinear wave equation in a separable Hilbert space H

$$\begin{aligned} &\frac{d^2u}{dt^2} + Au + M(u) = 0, \\ &u(0) = \phi, \quad u'(0) = \psi. \end{aligned} \tag{1.2.7}$$

where $A : D(A) \to H$ be strictly positive, linear, self-adjoint operator and $\phi \in D(A)$, $\psi \in D(A^{\frac{1}{2}})$ and $M : D(A^{\frac{1}{2}}) \to H$ be a single valued nonlinear map. He has proved the convergence of an approximate solution to the solution of (1.2.7). He has first introduced the nonlinearities M which are "reproducing" relative to a complete orthonormal sequence $\{u_i\}_1^\infty$ and obtained a finite system of the Faedo-Galerkin approximating ODE. Then he has shown that if $\{u_i\}_1^\infty$ are eigen functions of A, then the solution of the Faedo-Galerkin approximating ODE converges to that of (1.2.7) under some assumptions. His procedure may be considered as generalized separation of variables for a class of nonlinear wave equations. It can be extended to the inhomogeneous equation

$u'' + Au + M(u) = f$, as well as the parabolic equation $u' + Au + M(u) = f$. In [21], he has proved the global convergence of the Faedo-Galerkin approximations to nonlinear wave equations.

Miletta [72] have proved the convergence of the Faedo-Galerkin approximations to the solution of an abstract semi-linear evolution of the form

$$\frac{du(t)}{dt} + Au(t) = M(u(t)),$$
$$u(0) = \phi,$$

in a separable Hilbert space H where A is closed, positive definite, self-adjoint, linear operator with domain $D(A)$ dense in H. Also assumed is that A has a pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq ....$$

and a corresponding complete orthonormal system $\{u_i\}$ so that

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{i,j},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

These assumptions on A guarantee that $-A$ generates an analytic semigroup e^{-tA}. M is a continuous nonlinear operator from $D(A^\alpha) \to H$ for some α, $0 < \alpha < 1$, which satisfies the following Lipschitz condition on the balls in $D(A^\alpha)$: For each $c > 0$ there exists a constant $K(c)$ s.t.,

(a) $\|M(u)\| \leq K(c)$ for $u \in D(A^\alpha)$ with $\|A^\alpha u\| \leq c$,

(b) $\|M(u_1) - M(u_2)\| \leq K(c)\|A^\alpha(u_1 - u_2)\|$ for $u_i \in D(A^\alpha)$ with $\|A^\alpha u_i\| \leq c$ for $i = 1, 2$.

Extending the technique of Miletta [72], Bahuguna, Srivastava and Singh [16] have proved the global convergence of the Faedo-Galerkin approximation to the solution of

an integro-differential equation of the form

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)) + \int_0^t a(t-s)g(s, u(s))ds,$$
$$u(0) = \phi,$$

where A satisfy the same assumption but nonlinear maps f and g are defined from $[0,T] \times X_\alpha(T)$ into H, $X_\alpha(T) = C([0,T], D(A^\alpha))$ for $0 < \alpha < 1$ endowed with the supremum norm and the kernel a is in $L^p(0,T)$ for some $1 < p < \infty$.

Bahuguna [4, 5] have established the existence, uniqueness, continuation of solutions to the maximal interval of existence, and the global existence of a strong and classical solution of the abstract second order semi-linear equation,

$$\frac{d^2u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) + Bu(t) = f(t, u(t), \frac{du}{dt}(t)), \quad t > 0,$$
$$u(0) = x_0, \quad u'(0) = x_1, \tag{1.2.8}$$

under the suitable conditions.

Problem (1.2.8) for the particular case $L = -\Delta$, the n-dimensional Laplacian, is dealt with by Duvaut and Lions [31] and Glowinski, Lions and Tremolieres [34] in the context of some viscoelastic materials using the method of variational inequalities.

Sandefur [84] has considered (1.2.8) for a special case where the operators A and B are such that the equation can be written in the following factorized form,

$$\frac{d^2u}{dt^2} - (A_1 + A_2)\left(\frac{du}{dt}\right) + A_1A_2u = f(t, u), \quad t \in (0, T],$$
$$u(0) = x_0, \quad \frac{du}{dt}(0) = x_1. \tag{1.2.9}$$

Using the theory of semigroups together with the method of successive approximations, Sendefur [84] applied the results to various physical problems, for instance, the damped

and strongly damped semi-linear equations, the telegraph equation and the equation of motion for a thin panel.

Aviles and Sandefur [3] have modified the techniques of Sandefur [84] by changing the order of integration in the definition of the mild solution to (1.2.9) and applied the results to the Klein-Gordon equation, the von Karman equation and the vibrating beam equation.

Webb [95] and Ang and Dinh [2] have considered the following initial boundary value problem,

$$u_{tt} - \lambda \Delta u_t + f(u) = 0, \quad (x,t) \in \Omega \times (0,T), \;\; \lambda > 0,$$
$$u = 0, \quad (x,t) \in \partial\Omega \times [0,T), \quad u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \qquad (1.2.10)$$

where Ω is a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$. For $1 \leq N \leq 2$, (1.2.10) governs the motion of a linear Kelvin solid (a bar if $N = 1$, a plate if $N = 2$) subject to certain nonlinear elastic constraints. Webb [95] has studied the existence and the asymptotic behavior of a solution to (1.2.10) for $N = 1, 2, 3$. Ang and Dinh [2] have generalized the results of Webb [95] for (1.2.10) using the method of successive linearizations.

Engler, Neubrander and Sandefur [32] have used the theory of semigroups to prove the existence and uniqueness of a mild solution of (1.2.8). Balachandran and Park [17] has also proved the existence of a mild solution by using the theory of strongly continuous cosine families and the Schaefer fixed point theorem.

The Rothe Method, introduced by E. Rothe [83] in 1930, is an efficient theoretical tool for solving a broad scale of evolution equations. Unlike some abstract method for the analysis of the existence and uniqueness of a solution for an evolution equation, the Rothe method has a strong numerical aspects. The problem solved originally by

E. Rothe was very simple one. However, his method turned out to be a very efficient tool for obtaining the solution of substantially more complicated evolution problems, for example, linear and quasi-linear parabolic problems of the second order in n dimensions, a parabolic problem of arbitrary order, nonlinear problems, hyperbolic problems, the Stephen problem, integro-differential problems, mixed parabolic-hyperbolic problems, etc.

The Rothe method has been used and developed by many authors, e.g. O.A. Ladyženskaja [66], T.D. Ventcel, A. M. Iljin, A. S. Kalašnikov, O. A. Olejnik, S.I. Ibragimov, P.S. Mosolov and K. Rektorys [80, 82] in linear and quasi-linear parabolic problems. Nonlinear and abstract parabolic problems has been studied by J. Kačur [44, 45, 47], J. Nečas [74], A. G. Kartsatos, M. E. Parrott [51, 52] etc. linear and quasi-linear hyperbolic equations has been considered by J. Jerome, E. Martensen, M. Pultar [79], J. Kačur [46] etc. Many of the obtained results by the Rothe method were obtained as well by the other methods like the method of compactness, the theory of semigroups, the method of monotone operators, the Fourier transform, etc.

Two powerful theoretical tools, among many methods, are the theory of semigroups and theory of monotone operators for proving the existence and uniqueness results for a differential equation in an abstract space. The theory of semigroups of linear operators developed quite rapidly after the establishment of the theorem of Hille and Yosida in 1948 and by now it is an interesting mathematical subject with substantial applications to many fields, in particular, to partial differential equations. The books by Hille and Phillips [43], Ladas and Lakshmikantam [65], Kato [56], Pazy [77], Martin [70], Showalter [90] among several others are extensive treatises on the subject of semigroups of linear operators. One of the important results in the theory of semigroups of bounded linear operators is due to R.A. Phillips [78] for the case of a Hilbert space and its extension in a general Banach space is due to Lumer and Phillips [68]. This result

characterizes the infinitesimal generator of a contraction semigroup of bounded linear operators as an m-dissipative operator (see Pazy [77], p. 13-14).

The theory of monotone operators with the Rothe method has been incorporated for the first time by Nečas [74]. Nečas has studied the following nonlinear equation in a real Hilbert space H:

$$\frac{du}{dt}(t) + Au(t) = f(t), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{1.2.11}$$

where u is defined on $[0, T]$ into H, A is a nonlinear m-monotone operator defined on the domain $D(A) \subseteq H$ of A into H and f, defined on $[0, T]$ into H, is a continuous function of bounded variation.

Kartsatos and Zigler [53] have extended the results of Nečas for the semi-linear problem

$$\begin{aligned} &\frac{du}{dt}(t) + Au(t) = G(t, u(t)), \quad 0 < t \leq T, \\ &u(0) = u_0, \end{aligned} \tag{1.2.12}$$

considered in a Banach space X whose dual space is uniformly convex where A defined on the domain $D(A) \subseteq X$ into X is m-monotone and G defined on $[0, T] \times X$ into X satisfies a Lipschitz-like condition of the type

(A1) $\|G(t, x) - G(s, y)\| \leq \|\phi(t) - \phi(s)\| + L\|x - y\|$

with $\phi : [0, T] \to X$ a continuous function of bounded variation and $L \geq 0$ a constant.

In [47] Kačur has applied Rothe's Method to the nonlinear "implicit" integro-differential equation

$$\begin{aligned} &\frac{du}{dt}(t) + Au(t) = G(t, K(u)(t)), \quad \text{a.e.} \quad t \in (0, T), \\ &u = \phi \quad \text{on} \quad [-q, T], \quad q > 0. \end{aligned} \tag{1.2.13}$$

where u is defined on $[0,T]$ into a Hilbert space H, A is a nonlinear operator defined on a reflexive Banach space V into its dual V^*, the nonlinear map G is defined on $[0,T]\times H$ into H and K is a Volterra operator defined on $L_\infty([-q,T],H)$ into $L_\infty([-q,T],H)$. Kačur has proved the existence of a unique strong solution under the following hypotheses for a given Lipschitz continuous function $\phi : [-q,T] \to H$.

(A2) A is a coercive maximal monotone operator. Coercivity is assumed in the form

$$< Au, u > \;\geq\; \|u\|P(\|u\|) - C_1|u|^2 - C_2,$$

for all u in V where $\|.\|$ and $|.|$ are the respective norm of V and H, $< .,. >$ represents the usual duality product between V^* and V and $P(s) \to \infty$ as $s \to \infty$.

(A3) G satisfies a Lipschitz condition of the type

$$|G(t,u) - G(s,v)| \;\leq\; C[|t-s| + |t-s||u| + |u-v|],$$

for all $t, s \in [0,T]$ and all $u, v \in H$ where $C \geq 0$ is a constant.

(A4) The nonlinear Volterra operator K maps $\mathrm{Lip}(S_T, H)$ into $\mathrm{Lip}(S_T, H)$ where $S_T; = [-q,T]$ and satisfies

(i) $\|K(u) - K(v)\|_{C(S_T,H)} \;\leq\; C\|u-v\|_{C(S_T,H)}$,

for all $u, v \in C(S_T, H)$ where $C \geq 0$ is a constant; and

(ii) $|K(u)(t) - K(u)(s)| \;\leq\; |t-s|L(\|u\|_{C(S_T,H)})\left(1 + \|\frac{du}{dt}\|_{L_\infty(S_T,H)}\right)$,

for all $s, t \in S_T$, $s < t$, $u \in \mathrm{Lip}(S_T, H)$ and $L : \mathrm{R}_+ \to \mathrm{R}_+$ is continuous.

In the earlier studies on Rothe's method, we notice that one proves an *apriori* bound for difference quotients $\frac{u_j - u_{j-1}}{h}$ without proving any bounds for the discrete points u_j, hence required global Lipschitz conditions of the types (A1), (A2) and (A4) which has been considered by Kartsatos and Zigler [53] and Kačur [47]. Bahuguna modified Rothe's method which allows to weaken the global Lipschitz condition to a local Lipschitz condition plus a growth condition. Using the growth condition, he

has proved that the discrete points lie in a ball whose radius is independent of the discretization parameters and then using the local Lipschitz condition he has proved the bounds for the difference quotients.

Bahuguna and Raghavendra [8] have considered the abstract integro-differential equation

$$\frac{du}{dt}(t) + Au(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t), \quad t > 0,$$
$$u(0) = u_0, \tag{1.2.14}$$

in a Banach space X whose dual is uniformly convex where A is a nonlinear single valued operator from $D(A) \subset X$ into X, the mapping k is defined on $[0,t] \times X$ into X, the X-valued function f and the real-valued function a are defined on $[0,T]$. He has proved, using Rothe's method, the existence of a strong solution of (1.2.14) under the assumption that the operator A is m-monotone such that $(I+A)^{-1}$ is compact, mapping k is continuous in both the variables and satisfies the growth condition

$$\|k(t,x)\| \ \leq \ C_1\|x\| + C_2,$$

for all $t \in [0,T]$ and $x \in X$, where C_1, $C_2 \geq 0$ are the constants, the functions f and a are the continuous functions with bounded variations on $[0,T]$. The uniqueness is proved under an additional condition of Lipschitz continuity on $k(t,x)$ in x, a.e. $t \in [0,T]$.

Bahuguna [9] has also used Rothe's method to establish the existence of a unique strong solution of the nonlinear implicit abstract integro-differential equation

$$\frac{du}{dt}(t) + Au(t) = f(t,u(t),F(u)(t)), \quad \text{for a.e.} \quad t \in (0,T),$$
$$u = \phi \quad \text{on} \quad [-q,0], \quad \phi \in Lip([-q,0],H), \tag{1.2.15}$$

in a Hilbert space H.

The initial work using the theory of m-accretive operators to the quasi-linear evolution equations is due to Kartsatos [48]. Kartsatos [49] and Kartsatos and Parrot [50] have established the existence and uniqueness and used the Galerkin method for the approximation of solutions to the quasi-linear functional problem

$$u'(t) + A(t, u_t)u(t) = 0, \quad t \in [0, T), \quad u(0) = \phi,$$

in a Banach space X, where $A(t, \phi)v$ is m-accretive in $v \in D$ for every pair $(t, \phi) \in [0, T) \times C_0$, D is a subset of X, C_0 is a closed subset of the space of all continuous functions ϕ on $(-\infty, 0]$ into D with $\|\phi\|_\infty \leq r$ for some fixed $r > 0$ and $u_t \in C_0$, given by $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$. Murphy [73] has constructed a family of approximate solutions to the homogeneous quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = 0, \quad t \in (0, T], \quad u(0) = \phi. \tag{1.2.16}$$

He has shown that the approximate solutions converge to a *limit solution* and this limit solution becomes a unique solution. Kartsatos [48] established a theorem concerning the existence of a unique strong solution to (1.2.16) under the assumption that $A(t, u)v$ is Lipschitzian in t, u and m-accretive in v. In this paper, conditions on $A(t, u)v$ is motivated by Kartsatos [48].

For the study of the abstract quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = f(t, u(t)), \quad t \in (0, T], \quad u(0) = \phi, \tag{1.2.17}$$

we refer to T. Kato [61, 58, 60, 62], S. Kato [54], Amann [1] and references cited therein. The crucial assumption in these papers is that the linear operator $A(t, u)$, depending

on t and the unknown u, has the dual property of being the negative generator of a C_0-semigroup but not necessarily of an analytic semigroup and at the same time, a bounded operator on Y into X. Moreover, in [61, 58, 60, 62] there is an isomorphism S of Y onto X with the property that

$$SA(t,w)S^{-1} = A(t,w) + B(t,w),$$

where $B(t,w)$ is in the space $B(X)$ of all bounded linear operators from X into X whereas in [54] it is assumed that there is a family $\{S(t,w)\}$ of the isomorphisms of Y onto X such that

$$S(t,w)A(t,w)S(t,w)^{-1} = A(t,w) + B(t,w),$$

where $B(t,w) \in B(X)$ and $S(t,w)$ satisfies the Lipschitz-like condition

$$\|S(t,w_1) - S(s,w_2)\|_{Y,X} \leq \mu(|t-s| + \|w_1 - w_2\|_X),$$

where μ is a constant. Kato [61] has proved the existence, uniqueness and continuous dependence on the initial data of the solution of an abstract quasi-linear and has shown that these results are applicable to the different kinds of quasi-linear equations such as the symmetric hyperbolic systems of the first order, the wave equation, the Korteweg-de Vries equation, the Navier-Stokes and the Euler equation, a magnetohydrodynamics equation, the coupled Maxwell and the Dirac equations, etc. Kato [54] has proved the existence of a strong solution to (1.2.17) under the conditions on $A(t,u)$ and $f(t,u)$ for $(t,u) \in J \times W$ similar to that of Crandal and Sougandis [28]. Amann [1] has treated the various cases of (1.2.17) in interpolation spaces using the theory of analytic semigroups.

Bahuguna [4] has also proved the similar results by using Rothe's method for the abstract quasi-linear explicit integro-differential equation

$$\frac{du}{dt}(t) + A(u(t))u(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t),$$

for $0 < t < T$ with the initial value u_0 in real reflexive Banach space X whose dual is uniformly convex, $A(u)$ is a linear operator in X for each u in an open subset W of Y where Y is also a real reflexive Banach space which is continuously and compactly embedded in X, a is a real-valued and f is a Y- valued functions defined on $[0, T]$ and k is a Y- valued map defined on $[0, T] \times W$. Using the semigroup theory together with the contraction mapping theorem Bahuguna [5] has established the existence of a unique classical solution to quasi-linear explicit integro-differential equation in the Banach space X,

$$\frac{du}{dt}(t) + A(t, u(t))u(t) = \int_0^t a(t-s)k(s, u(s))ds + f(t),$$
$$u(0) = u_0,$$

where $A(t, u)$ is a linear operator in X for each u in an open subset W of Y, maps a, f, k and spaces X, Y have same properties as in [4]. Also using similar techniques of paper [4], we [10] have established the existence, uniqueness and continuous dependence of a strong solution to the quasi-linear "implicit" integro-differential equations

$$\frac{du}{dt}(t) + A(u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0,$$

in a reflexive Banach space whose dual is uniformly convex.

1.3 Preliminaries

In this section, we mention some of the results required for the analysis in the ensuing chapters. We mostly mention these results without proofs with proper references.

1.3.1 Semigroups

Definition 1.3.1 Let X be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a *semigroup of bounded linear operator on* X if

(i) $T(0) = I$, where I is the identity operator on X.

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

The linear operator A defined by

$$D(A) = \left\{ x \in X \ : \ \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad \text{exists} \right\}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} = \left. \frac{d^+T(t)x}{dt} \right|_{t=0} \quad \text{for} \quad x \in D(A)$$

is the *infinitesimal generator* of the semigroup $T(t)$, $D(A)$ is the domain of A.

Example 1.3.2 Let $T(t) : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$T(t) = e^{\lambda t} x_0, \quad t \geq 0, \quad x_0 \in \mathbf{R}.$$

Then $T(t)$ is a semigroup of bounded linear operators on $\mathbf{R}$ and its infinitesimal generator is $A = \lambda I$.

Example 1.3.3 Let $T(t) : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be such that

$$T(t) = e^{tB} x_0, \quad t \geq 0, \quad x_0 \in \mathbf{R}^n,$$

where $B = [b_{ij}]$ is an $n \times n$ matrix, b_{ij} are the constants. Then $T(t)$ is a semigroup of bounded linear operators on $X = \mathbf{R}^n$ and its generator is $A = B$.

Example 1.3.4 Let $T(t) : X \longrightarrow X$ be given by

$$T(t) = e^{tB}x_0, \quad t \geq 0, \quad x_0 \in \mathbf{R},$$

where $B : X \longrightarrow X$ is a bounded linear operator. Then $T(t)$ is a semigroup of bounded linear operators on X. and its generator is $A = B$.

Definition 1.3.5 The semigroup of bounded linear operators, $T(t)$ is *uniformly continuous* if

$$\lim_{t \downarrow 0} \|T(t) - I\| = 0.$$

We note that the semigroup $T(t)$ in each of the above examples is uniformly continuous. In fact, we have the following result.

Theorem 1.3.6 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. Furthermore, the semigroup $T(t)$ generated by a bounded linear operator $A : X \longrightarrow X$ is unique and*

(a) There exists $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.

(b) The map $t \mapsto T(t)$ is differentiable in norm and

$$\frac{dT(t)}{dt} = AT(t) = T(t)A.$$

For applications to partial differential equations, we can not expect the infinitesimal generator of a semigroup to be a bounded linear operator. Therefore, the class of uniformly continuous semigroups is of little use for this purpose. Thus, we need to consider other type of semigroups.

Definition 1.3.7 A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a *strongly continuous* semigroup of bounded linear operator if

$$\lim_{t \downarrow o} T(t)x = x \quad \text{for every} \quad x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a *semigroup of class* C_0 or simply a C_0-*semigroup.*

Example 1.3.8 In the Banach space $X = C[0,1]$ of continuous functions with sup-norm, define the family of operators $\{T(t)\}$, $t \geq 0$ by the formula

$$T(t)x(\xi) = x[\xi/(1+t\xi)], \quad x \in X, \quad \xi \in [0,1].$$

Then $\{T(t)\}$, $t \geq 0$, is a C_0-semigroup of operators in X.

If $T(t)$ is a C_0-semigroup, then there are constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for} \quad 0 \leq t < \infty.$$

Also. for every $x \in X$, the map $t \mapsto T(t)x$ is a continuous function from $\mathbf{R}_0^+$ (the nonnegative real line) into X (cf. §1.2 in Pazy [77]).

We also have the following main result for C_0-semigroups.

Theorem 1.3.9 *Let $T(t)$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

(a) For $x \in X$,

$$\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x ds = T(t)x.$$

(b) For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

(c) For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

(d) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)d\tau.$$

(cf. Theorem 1.2.4 in Pazy [77].)

Let $T(t)$ be a C_0-semigroup, then there are constants $\omega \geq 0$ and $M \geq 1$ such that $||T(t)|| \leq Me^{\omega t}$ for $t \geq 0$. If $\omega = 0$, then $T(t)$ is called *uniformly bounded semigroup* and if moreover $M = 1$, then $T(t)$ is called *C_0-semigroup of contractions.*

Example 1.3.10 In the Banach space $X = C[0, \infty)$ of continuous, bounded functions on $[0, \infty)$ with sup-norm, define the family of translation operators $\{T(t)\}$, $t \geq 0$, by the formula

$$T(t)x(\xi) = x(t + \xi), \quad x \in X, \quad \xi \geq 0.$$

Then $\{T(t)\}$, $t \geq 0$, is a C_0-semigroup of contraction of operators in X.

For a linear operator A not necessarily bounded, in X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

One of main results in the theory of semigroups of bounded linear operators is the Hille-Yosida theorem which was established in 1948 by Hille [42] and Yosida [96]. This result establishes a characterization of the infinitesimal generator of a C_0-semigroup of contraction.

Theorem 1.3.11 *(Hille-Yosida). A linear (unbounded) operator A is the infinitesimal generator of a C_0-semigroup of contractions $T(t)$, $t \geq 0$ if and only if*

(i) A is closed and $D(A)$ is dense in X,

(ii) the resolvent set $\rho(A)$ of A contains $[0, \infty)$ and for every $\lambda > 0$

$$||R(\lambda : A)|| \leq \frac{1}{\lambda}.$$

We note that if $T(t)$ is a C_0-semigroup with A its infinitesimal generator such that for some $\omega \geq 0$, $\|T(t)\| \leq e^{\omega t}$, then $S(t) = e^{-\omega t}T(t)$ is a C_0-semigroup of contractions and its infinitesimal generator is $A - \omega I$.

The Hille-Yosida theorem gives us a characterization of the infinitesimal generator of a C_0-semigroup of contractions. Another characterization of such operators is given by the Lumer-Phillips theorem which we state below.

Now, we consider the notion of the dissipative operators. To define the dissipative operators we require duality map defined from X into the dual space X^* of X, given by

$$F(x) = \left\{x^* \in X^* \ : < x^*, x > = \ \|x\|^2 = \|x^*\|_*^2\right\},$$

where $< .,. >$ denotes the duality paring between the dual space X^* of X and X. Clearly, duality map is a multi-valued or set valued map in general. From the Hahn-Banach theorem it follows that $F(x) \neq \emptyset$ for every $x \in X$. In particular, when X is a Hilbert space then the duality map is single-valued.

Definition 1.3.12 *A linear operator A is called dissipative if for every $x \in D(A)$ there exists a $x^* \in F(x)$ such that*

$$Re < x^*, Ax > \ \leq \ 0.$$

A dissipative operator A for which $R(I - \alpha A) = X$, for every $\alpha > 0$ is called m-dissipative. If operator A is dissipative the $-A$ is accretive or monotone operator.

The following theorem gives the characterization of the dissipative operators.

Theorem 1.3.13 *A linear operator A is dissipative if and only if*

$$\lambda\|x\| \;\leq\; \|(\lambda I - A)x\|$$

for all $x \in D(A)$ and $\lambda > 0$.

Theorem 1.3.14 *(Lumer and Phillips) Let A be a linear operator with dense domain $D(A)$ in X. Then we have the following.*

(a) if A is dissipative and there exists a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$ of $(\lambda_0 I - A)$ is X, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.

(b) if A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(X)$, $Re < x^*, Ax > \;\leq\; 0$.

(cf. Theorems 1.4.2 and 1.4.3, resepectively, in Pazy [77].)

As pointed out earlier that what we know is the operator A associated with some partial differential operator and the problem is to find the semigroup generated by it. For a bounded linear operator B, we already know that the semigroup is uniformly continuous and given by e^{tB}. The main task is to find the representation of the semigroup generated by an unbounded linear operator, if there is any. For this purpose we use the complex operator calculus. For a bounded linear operator B we have another representation given by the following theorem.

Theorem 1.3.15 *Let B be a bounded linear operator. If $\gamma > \|B\|$, then*

$$e^{tB} = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}(\lambda I - B)^{-1} d\lambda. \tag{1.3.1}$$

The convergence in (1.3.1) is in the uniform operator topology and uniformly in t on bounded intervals.

(cf. Theorem 1.7.1 in Pazy [77].)

The following theorem gives the sufficient conditions for an operator A to be the infinitesimal generator of a C_0-semigroup and a similar representation as in (1.3.1).

Theorem 1.3.16 *Let A be densely defined operator in X satisfying the following conditions.*

(i) For some $0 < \delta < \pi/2$, $\rho(A) \supset \Sigma_\delta = \{\lambda : |arg\lambda| < (\pi/2) + \delta\} \cup \{0\}$.

(ii) There exists a constant M such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}$$

for $\lambda \in \Sigma_\delta$ with $\lambda > 0$.

Then, A is the infinitesimal generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq C$ for some positive constant C and

$$T(t) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A)^{-1} d\lambda$$

where Γ is a smooth curve in Σ_δ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\pi/2 < \theta < \pi/2 + \delta$. The integral converges for $t > 0$ in the uniform operator topology.

(cf. Theorem 1.7.7 in Pazy [77].)

We are more interested in the special kind of semigroups known as analytic semigroup from the application point of view. Therefore we now consider the possibility of extending the domain of the parameter to a region of the complex plane that includes the negative real axis.

Definition 1.3.17 Let $\Delta = \{z \in \mathbf{C} \ : \ \varphi_1 < \arg z < \varphi_2, \quad \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \Delta$ is an *analytic semigroup* in Δ if

(i) $z \to T(z)$ is analytic in Δ;

(ii) $T(0) = I$ and $\lim_{z\to 0} T(z)x = x$ for every $x \in X$;

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

A semigroup $T(t)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

If $T(t)$ is a C_0-semigroup which can be extended to an analytic semigroup in some sector then the semigroup $e^{\omega t}T(t)$ can also be extended to an analytic semigroup in some sector. Therefore we may restrict ourselves in most of the cases to uniformly bounded C_0-semigroups. For convenience, we suppose that $0 \in \rho(A)$ where A is the infinitesimal generator of the semigroup $T(t)$. This may also be achieved by multiplying the uniformly bounded semigroup $T(t)$ by $e^{-\epsilon t}$ for $\epsilon > 0$.

We assume that the densely defined operator A in X satisfies

$$\rho(A) \supset \sum = \left\{\lambda \ : \ |\arg \lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\} \tag{1.3.2}$$

for some $0 < \delta < \frac{\pi}{2}$, and

$$\|R(\lambda : A)\| \ \le \ \frac{M}{|\lambda|} \quad \text{for} \quad \lambda \in \sum, \quad \lambda \neq 0. \tag{1.3.3}$$

It follows that if A is densely defined operator, A satisfies (1.3.2) and (1.3.3), then the semigroup generated by A can be extended to an analytic semigroup in the sector $\Delta_\delta = \{z \ : \ |\arg z| < \delta\}$ and every closed sub-sector $\overline{\Delta}_{\delta'} = \{z \ : \ |\arg z| \le \delta' < \delta\}$, $\|T(z)\|$ is uniformly bounded. In fact, we have the following theorem.

Theorem 1.3.18 *Let $T(t)$ be a uniformly bounded C_0-semigroup. Let A be the infinitesimal generator of $T(t)$ and assume that $0 \in \rho(A)$. Then the following are equivalent.*

(a) $T(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z \ : \ |arg\ z| < \delta\}$ and $\|T(t)\|$ is uniformly bounded in every closed sub-sector $\overline{\Delta_{\delta'}}$ of Δ_δ, $\delta' < \delta$ of Δ_δ.

(b) There exists a constant C such that for every $\sigma > 0$, $\tau \neq 0$,

$$\|R(\sigma + i\tau : A)\| \ \leq \ \frac{C}{|\tau|}.$$

(c) There exist $0 < \delta < \pi/2$ and $M > 0$ such that

$$\rho(A) \supset \Sigma = \left\{\lambda \ : \ |arg\ \lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$$

and

$$\|R(\lambda : A)\| \ \leq \ \frac{M}{|\lambda|} \quad for \quad \lambda \in \Sigma, \quad \lambda \neq 0.$$

(d) $T(t)$ is differentiable for $t > 0$ and there exists a constant C such that

$$\|AT(t)\| \ \leq \ \frac{C}{t}, \quad t > 0.$$

(cf. Theorem 2.5.2 in Pazy [77]).

1.3.2 Fractional Powers of Operators

For an operator for which $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, it is possible to define fractional power of A. More precisely, we suppose that A is a densely defined closed linear operator satisfying

$$\rho(A) \supset \sum\nolimits^{+} = \{\lambda \ : \ 0 < \omega < |\arg \lambda| \leq \pi\} \cup V, \tag{1.3.4}$$

where V is a neighborhood of zero, and

$$\|R(\lambda : A)\| \ \leq \ \frac{M}{1 + |\lambda|} \quad \text{for} \quad \lambda \in \sum\nolimits^{+}. \tag{1.3.5}$$

If $M = 1$ and $\omega = \pi/2$ then $-A$ is the infinitesimal generator of a C_0-semigroup. If $\omega < \pi/2$ then, $-A$ is the infinitesimal generator of an analytic semigroup.

For an operator satisfying (1.3.4) and (1.3.5) and $\alpha > 0$ we define

$$A^{-\alpha} = \frac{1}{2\pi i}\int_C z^{-\alpha}(A - zI)^{-1}dz, \tag{1.3.6}$$

where C is a path in the resolvent set A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\omega < \theta < \pi$, avoiding the negative real axis and the origin and $z^{-\alpha}$ is taken to be positive for real positive values of z. The integral (1.3.6) converges in the uniform operator topology for every $\alpha > 0$ and thus defines a bounded linear operator $A^{-\alpha}$.

We observe that for $\alpha, \beta \geq 0$

$$A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta}$$

and there exists a constant C such that

$$\|A^{-\alpha}\| \leq C \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

For A satisfying (1.3.4) and (1.3.5) with $\omega < \pi/2$, for $\alpha > 0$, we have

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

For $\alpha = 0$, $A^{\alpha} = I$.

For the main application of the analytic semigroups, we require the following theorem.

Theorem 1.3.19 *Let $-A$ be infinitesimal generator of analytic semigroup $T(t)$. If $0 \in \rho(A)$, then we have the following.*

(a) $T(t) : X \to D(A^{\alpha})$ for every $t > 0$ and $\alpha \geq 0$.

(b) For every $X \in D(A^\alpha)$, we have $T(t)A^\alpha x = A^\alpha T(t)x$.

(c) For every $t > 0$ the operator $A^\alpha T(t)$ is bounded and

$$\|A^\alpha T(t)\| \;\leq\; M_\alpha t^{-\alpha} e^{-\delta t}$$

for some constants M_α, depending on α, and $\delta > 0$.

(d) For $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$,

$$\|T(t)x - x\| \;\leq\; C_\alpha t^\alpha \|A^\alpha x\|.$$

(cf. Theorem 2.6.13 in Pazy [77].)

Corollary 1.3.20 *Let A be the infinitesimal generator of an analytic semigroup $T(t)$. If $f \in L^1(0,T;X)$ is locally Holder continuous on $(0,T]$, then for every $x \in X$ the initial value problem*

$$\frac{du(t)}{dt} = Au(t) + f(t),$$
$$u(0) = x,$$

has a unique solution $u \in C((0,T] : X)$ given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

(cf. Corollary 4.3.3 in Pazy [77].)

1.3.3 Evolution Systems

Definition 1.3.21 *A two parameter family A two parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq T$, on X is called an* evolution system *if the following two conditions are satisfied:*

(i) $U(s,s) = I$ *and* $U(t,r)U(r,s) = U(t,s)$, $0 \le s \le r \le t \le T$.

(ii) The map $(t,s) \to U(t,s)$ *is strongly continuous for* $0 \le s \le t \le T$.

Definition 1.3.22 *Let X be a Banach space. A family $\{A(t)\}_{t\in[0,T]}$ of infinitesimal generators of C_0-semigroups S_t on X is called* stable *if there are constants $M \ge 1$ and ω (called the stability constants) such that*

$$\rho(A(t)) \supset]\omega, \infty[\quad \text{for} \quad t \in [0,T]$$

and

$$\left\| \prod_{j=1}^{k} R(\lambda : A(t_j)) \right\| \le M(\lambda - \omega)^{-k} \quad \text{for} \quad \lambda > \omega$$

and every finite sequence $0 \le t_1 \le t_2, \cdots, t_k \le T$, $k = 1, 2, \cdots$.

Let X and Y be two Banach Spaces such that Y is densely and continuously embedded in X, i.e., Y is a dense subset of X and there is a constant C such that

$$\|w\|_X \le C\|w\|_Y \quad \text{for} \quad w \in Y.$$

The subspace Y of X is an *invariant* subspace of a linear operator $S : D(S) \subset X \to X$ if $S : D(S) \cap Y \to Y$.

Let $S_t(s)$, $s \ge 0$, be C_0-semigroup generated by $A(t)$, $t \in [0,T]$. A subset Y of X called $A(t)$-*admissible* if Y is an invariant subspace of operator $S_t(s)$, $s \ge 0$ and the restriction $\tilde{S}_t(s)$ of $S_t(s)$ to Y is a C_0-semigroup on Y. Moreover, $\tilde{A}(t)$, the part of $A(t)$ is, in this case, the infinitesimal generator of the semigroup $\tilde{S}_t(s)$ on Y. We make the following assumptions.

(H1) $\{A(t)\}_{t\in J}$ is a stable family with stability constant M, ω.

(H2) Y is $A(t)$-admissible for $t \in J$ and the family $\{\tilde{A}(t)\}_{t\in J}$ of parts $\tilde{A}(t)$ of $A(t)$ in Y, is a stable family in Y with stability constant $\tilde{M}$, $\tilde{\omega}$.

(H3) For $t \in J$, $D(A(t)) \supset Y$, $A(t)$ is a bounded operator from Y into X and $t \rightarrow A(t)$ is continuous in the $B(Y, X)$ norm $||.||_{Y\rightarrow X}$.

Theorem 1.3.23 *Let $A(t)$, $0 \leq t \leq T$, be the infinitesimal generator of a C_0-semigroup $S_t(s)$, $s \geq 0$, on X. If the family $\{A(T)\}_{t\in[0,T]}$ satisfies the conditions (H1)-(H2), then there exists a unique evolution system $U(t, s)$, $0 \leq s \leq t \leq T$, in X satisfying*

$$(E_1) \quad ||U(t,s)|| \;\leq\; Me^{\omega(t-s)} \quad \textit{for} \quad 0 \leq s \leq t \leq T.$$

$$(E_2) \quad \frac{\partial^+}{\partial t}U(t,s)w\bigg|_{t=s} = A(s)w \quad \textit{for} \quad w \in Y, \quad 0 \leq s \leq T.$$

$$(E_3) \quad \frac{\partial}{\partial s}U(t,s)w \;=\; -U(t,s)A(s)w \quad \textit{for} \quad \omega \in Y, \quad 0 \leq s \leq t \leq T.$$

Where the derivative from the right in (E_2) and the derivative in (E_3) are in the strong sense in X.

(cf. Theorem 5.3.1 in Pazy [77].)

Theorem 1.3.24 *Let $\{A(t)\}_{t\in[0,T]}$ satisfy the condition of Theorem (1.3.23) and let $U(t, s)$, $0 \leq s \leq t \leq T$ be the evolution system given in Theorem (1.3.23). If*

$$(E_4) \quad U(t,s)Y \subset Y \quad \textit{for} \quad 0 \leq s \leq t \leq T,$$

(E_5) *for* $v \in Y$, $U(t,s)v$ *is continuous in* Y *for* $0 \leq s \leq t \leq T$

and $f \in C([s,T] : Y)$, *then for every* $v \in Y$ *the initial value problem*

$$\frac{du(t)}{dt} = A(t)u(t) + f(t) \quad \text{for} \quad 0 \leq s \leq t \leq T,$$
$$u(s) = v, \tag{1.3.7}$$

possesses a unique Y*-valued solution* u *given by*

$$u(t) = U(t,s)v + \int_s^t U(t,r)f(r)dr.$$

(cf. Theorem 5.5.2 in Pazy [77].)

Theorem 1.3.25 *(Contraction Mapping Theorem) Let* (X,d) *be a (nonempty) complete metric space and let* $F : X \to X$ *be a function. If there exists* $0 < r < 1$ *such that* $d(F(x),F(y)) \leq rd(x,y)$ *for all* $x, y \in X$, *then* F *has a unique fixed point in* X, *i.e., there exists a unique* $x \in X$ *such that* $Fx = x$.

Theorem 1.3.26 *(Gronwall's Inequality) Let* C *be a non-negative constant and Let* u *and* v *be nonnegative continuous function on some interval* $t_0 \leq t \leq t_0 + a$ *satisfying*

$$u(t) \leq C + \int_{t_0}^t u(s)v(s)ds, \quad t \in [t_0, t_0 + a],$$

then the inequality

$$u(t) \leq C\exp\left[\int_{t_0}^t v(s)ds\right], \quad t \in [t_0, t_0 + a]$$

holds.

Lemma 1.3.27 *(cf. Theorem 5.6.7 in Pazy [77]) Let $\varphi(t,s) \geq 0$ be continuous on $0 \leq s < t \leq T$. If there are positive constants A, B, α such that*

$$\varphi(t,s) \;\leq\; A + B\int_s^t (t-\sigma)^{\alpha-1}\varphi(\sigma,s)d\sigma \quad for \quad 0 \leq s < t \leq T, \tag{1.3.8}$$

then there is a constant K such that $\varphi(t,s) \leq AK$ for $0 \leq s < t \leq T$.

Proof. We have a well-known identity

$$\int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{\beta-1}d\tau = (t-s)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.3.9}$$

Iterating (1.3.8) $n-1$ times using the identity (1.3.9) and estimating $t-s$ by T we find

$$\varphi(t,s) \;\leq\; A\sum_{j=0}^{n-1}\left(\frac{BT^{\alpha}}{\alpha}\right)^j + \frac{(B\Gamma(\alpha))^n}{\Gamma(n\alpha)}\int_s^t (t-\sigma)^{n\alpha-1}\varphi(\sigma,s)d\sigma.$$

Choosing n sufficiently large so that $n\alpha > 1$ and estimating $(t-\sigma)^{n\alpha-1}$ by $T^{n\alpha-1}$, we get

$$\varphi(t,s) \;\leq\; C_1 + C_2\int_s^t \varphi(\sigma,s)d\sigma,$$

where

$$C_1 = A\sum_{j=0}^{n-1}\left(\frac{BT^{\alpha}}{\alpha}\right)^j \quad \text{and} \quad C_2 = \frac{(B\Gamma(\alpha))^n}{\Gamma(n\alpha)}T^{n\alpha-1}.$$

Gronwall's inequality implies that

$$\varphi(t,s) \leq C_1 e^{C_2(t-s)} \;\leq\; C_1 e^{C_2 T} \leq AK.$$

Since C_1 and C_2 do not depend on s this estimate hold for $0 \leq s < t \leq T$.

Let X and Y are reflexive Banach spaces and Y is continuously and densely embedded in X. Let $N(X,\beta)$ be set of densely defined linear operators C in X such that such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I+\lambda C)$ is one to one with a bounded inverse defined everywhere on X and

$$\|(I+\lambda C)^{-1}\|_X \;\leq\; (1-\lambda\beta)^{-1}$$

where I is identity operator.

Lemma 1.3.28 *Let $S : Y \to X$ be a linear isometric isomorphism, $C \in N(X, \beta)$, $Y \subseteq D(C)$, domain of C, $P \in B(X)$, the space of all bounded linear operators on X and $SC = CS + PS$. Set $\theta = \beta + \|P\|_X$. Then for every $y \in X$ and $\lambda > 0$ such that $\lambda\theta < 1$, the problems*

$$x + \lambda Cx = y$$

and

$$\hat{x} + \lambda(C\hat{x} + P\hat{x}) = y$$

have unique solutions x and $\hat{x}$ in X. Moreover

$$\|x\|_X \leq (1 - \lambda\theta)^{-1}\|y\|_X, \quad \|\hat{x}\|_X \leq (1 - \lambda\theta)^{-1}\|y\|_X,$$

and if $y \in Y$, then $x \in Y$ and

$$\|x\|_Y \leq (1 - \lambda\theta)^{-1}\|y\|_Y.$$

(cf. Lemma 2 [28].)

The operator A is called m-monotone if A satisfies the condition

$$\|u - v + \alpha(Au - Av)\| \geq \|u - v\| \quad \text{for every} \quad u, v \in D(A), \quad \alpha > 0 \qquad (1.3.10)$$

and the range $R(I + \alpha A) = X$ for $\alpha > 0$. For and m-monotone operator A, let us introduce the following sequences operators

$$J_n = (I + n^{-1}A)^{-1}, \quad A_n = AJ_n = n(I - J_n) \quad \text{for} \quad n = 1, 2, \cdots,$$

where AJ_n denotes the composition of the two maps A and J_n. The J_n and A_n are defined everywhere on X.

Lemma 1.3.29 *Let A be m-monotonic. J_n and A_n are uniformly Lipschitz continuous, with*

$$\|J_nx - J_ny\| \leq \|x-y\|, \quad \|A_nx - A_ny\| \leq 2n\|x-y\|,$$

where $2n$ may be replaced by n if X is a Hilbert space.

Lemma 1.3.30 *Let A be m-monotonic. The A_n are also monotonic. Furthermore, we have*

$$\|A_nu\| \leq \|Au\| \quad for \quad u \in D(A).$$

Lemma 1.3.31 *Let X be a Banach space with uniformly convex dual X^*. Let $A : D(A) \subset X \to X$ be m-monotone. Then we have the following.*
(a) If $u_n \in D(A)$, $n = 1, 2, \cdots$, $u_n \to u$ in X and if $\|Au_n\|$ are bounded then $u \in D(A)$ and $Au_n \rightharpoonup Au$. (Here $\rightharpoonup$ means weak convergence in X.)
(b) If $x_n \in X$, $n = 1, 2, \cdots$, $x_n \to u$ in X and if $\|A_nx_n\|$ are bounded, then $u \in D(A)$ and $A_nx_n \rightharpoonup Au$.
(c) $A_nu \rightharpoonup Au$ if $u \in D(A)$.

(cf. Lemma 2.2, 2.3 and 2.5, respectively, in Kato [56].)

1.3.4 Approximate Solutions

Now consider the abstract Cauchy problem

$$\frac{du}{dt}(t) + Au(t) = 0, \quad 0 < t \leq T, \quad u(0) = u_0, \tag{1.3.11}$$

where $A : D(A) \subset X \to X$ is m-monotone. Consider the approximate equations

$$\frac{du_n}{dt}(t) + A_nu_n(t) = 0, \quad u_n(0) = u_0. \tag{1.3.12}$$

It is proved by Kato [56] that if $u_0 \in D(A)$ then

$$\|u_n(t)\| \le C, \quad \|u'(t)\| \le C,$$

for all $n = 1, 2, \cdots$ where C is a constant independent of n and the strong limit $u(t) = \lim_{n\to\infty} u_n(t)$ exists uniformly on $[0, T]$ with u being Lipschitz continuous on $[0, T]$ and $u(0) = u_0$. Furthermore, Kato [56] has proved that $u(t) \in D(A)$ for $t \in [0, T]$ and $Au(t)$ is bounded and weakly continuous.

The main results of Kato [56], we are interested in are the following lemmas.

Lemma 1.3.32 *For each $f \in X^*$, $< u(t), f >$ is continuously differentiable on $[0, T]$ with*

$$\frac{d}{dt} < u(t), f > = - < Au(t), f > .$$

Proof: Since $u_n(t)$ satisfies (1.3.12), we have

$$< u_n(t), f > = < u_0, f > - \int_0^t < A_n u_n(s), f > ds.$$

Since $u_n(t) \to u(t)$ and $Au_n(t) \rightharpoonup Au(t)$ and $| < A_n u_n(t), f > | \le C\|f\|$, we get

$$< u(t), f > = < u_0, f > - \int_0^t < Au(s), f > ds. \tag{1.3.13}$$

Continuity of the integrand gives the required result.

1.3.5 Bochner Integral

Definition 1.3.33 Let $(S, \mathcal{B}, m)$ be a measure space and let u be a mapping defined on S with values in a Banach space X. u is called *weakly $\mathcal{B}$ measurable* if for any $f \in X^*$ the scalar function $s \mapsto < u(s), f >$ on S is $\mathcal{B}$-measurable. The function u is called *finitely-valued* if it is constant $\neq 0$ on each of a finite number of disjoint $\mathcal{B}$-measurable subsets B_i with $m(B_i) < \infty$ and $u \equiv 0$ on $S - \cup_i B_i$. A function u is said to be *strongly*

$\mathcal{B}$-measurable if there exists a sequence of finitely-valued functions strongly convergent to u m-a.e. on S.

Definition 1.3.34 The function u is said to be separably-valued if its range $\{u(s) : s \in S\}$ is separable, i.e., its range is a closure of a countable set. The function u is m-almost separably-valued if there exists a $\mathcal{B}$-measurable set B_0 of m-measure zero such that $\{u(s) : s \in S - B_0\}$ is separable.

From a result due to Pettis (cf. Yosida [96], p. 131), it follows that u is strongly $\mathcal{B}$-measurable if and only if it is weakly $\mathcal{B}$-measurable and m-almost separably-valued.

Let u be a finitely-valued function defined on a measure space $(S, \mathcal{B}, m)$ with values in a Banach space X. Let u be equal to $u_i \neq 0$ on $B_i \in \mathcal{B}$, $i = 1, 2, \cdots, n$, where B_i's are disjoint and $m(B_i) < \infty$ for $i = 1, 2, \cdots, n$ and $x \equiv 0$ on $(S - \cup_{i=1}^{n} B_i)$. Then we can define the m-integral $\int u(s)m(ds)$ of u over S by

$$\sum_{i=1}^{n} u_i m(B_i).$$

With the help of a limiting process, we can define the m-integral of more general functions. More precisely we have the following definition.

Definition 1.3.35 A function u defined on a measure space $(S, \mathcal{B}, m)$ with values in a Banach space X is said to be Bochner m-integrable, if there exists a sequence of finitely-valued functions $\{u_n\}$ which strongly converges to u m-a.e. in such a way that

$$\lim_{n \to \infty} \int_S \|u(s) - u_n(s)\| m(ds) = 0.$$

For any set $B \in \mathcal{B}$, the Bochner m-integral of u over B is defined by

$$\int_B u(s)m(ds) = \lim_{n \to \infty} \int_S \chi_B(s) u_n(s) m(ds),$$

where χ_B is the characteristic function of B, i.e., $\chi_B(s) = 1$ if $s \in B$ and it is equal to zero otherwise.

From a result due to Bochner (cf. Yosida [96], p. 133), we have that a strongly $\mathcal{B}$-measurable function u is Bochner m-integrable if and only if $||u||$ is m-integrable.

Lemma 1.3.36 *$Au(t)$ is Bochner integrable and $u(t)$ is an indefinite integral of $-Au(t)$. The strong derivative $du(t)/dt$ exists almost everywhere and equals $-Au(t)$.*

Proof: Let X_0 be the smallest closed linear subspace of X containing all the values of the $A_n u_n(t)$ for $t \in [0,T]$ and $n = 1, 2, \cdots$. Since $A_n u_n(t)$ are continuous, X_0 is separable. Since $A_n u_n(t) \rightharpoonup Au(t)$ and since X_0 is weakly closed it follows that $Au(t) \in X_0$. Thus $Au(t)$ is separably-valued. Since it is weakly continuous it is strongly measurable (cf. Yosida [96], p.131) and being bounded, it is Bochner integrable. From (1.3.13) we have that $u(t)$ is an indefinite integral of $-Au(t)$. For the existence of the strong derivative of $u(t)$ almost everywhere on $[0,T]$ and its being equal to $-Au(t)$ almost everywhere on $[0,T]$ we refer to Yosida [96], p. 134.

(cf. Lemma 4.6 in Kato [56].)

1.4 An Outline of Dissertation

In this section we briefly describe the analysis carried out in the ensuing chapters.

In Chapter 2, we consider the Sobolev type evolution equation in a separable Hilbert space and study the approximations of solutions. We first consider an associated integral equation and a sequence of approximate integral equations using the projection operators. Then we establish the existence of a unique solution to every approximate integral equation using the contraction mapping theorem. Next, after proving some estimates for the solution of the approximate integral equation, we prove the convergence of the solution of the approximate integral equation to the solution of

the associated integral equation. Finally, we consider the Faedo-Galerkin approximations of the solution and prove some convergence results. This work is an extension of a paper by the author [11] (with Bahuguna).

In Chapter 3, we consider a strongly damped semi-linear wave equation and reformulated it as a second order semi-linear evolution equation. Then we study the approximation of the solution to the second order semi-linear evolution equation in a separable Hilbert space by using the similar techniques which are used in Chapter 2. First, with the help of a pair of associated integral equations and projection operators, we consider the pair of approximate integral equations and prove the existence and uniqueness of a solution. We then establish the convergence of the pair of approximate integral equations to the pair of associated integral equations and the limit of the solution of the pair of approximate integral equations be the solution of the pair of associated integral equations. Finally, we consider the Faedo-Galerkin approximations of the solution and proved the convergence results. These results include the results of a paper by the author [12] (with Bahuguna).

In the next two Chapters, we study a class of second order semi-linear integro-differential equations arising in the study of viscoelastic material with memory.

In Chapter 4, we consider an abstract second order semi-linear integro-differential equation in a Banach space and first prove the existence and uniqueness of a local classical solution with the help of the semigroup theory and the contraction mapping theorem. Further, under some additional growth conditions on the nonlinear maps, we analyze the continuation of this solution, the maximum interval of existence and the global existence. This work is an extension of a paper by the author [91] (with Bahuguna).

In Chapter 5, we consider a strongly damped semi-linear integro-differential equa-

tion and reformulate it as a second order semi-linear integro-differential equation in a separable Hilbert space and study the convergence of the approximate solution using similar techniques which are used in Chapters 2 and 3. These results include the results of a paper by the author [13] (with Bahuguna).

In Chapter 6, we consider the first order quasi-linear implicit integro-differential equation in a Banach space. First, we prove the existence, uniqueness and continuous dependence on the initial data of a strong solution by using the method of semi-discretization in time in which we discretize the time axis and replace the time derivatives by the correspond difference quotients. Then we show that these discrete points lies in a ball whose radius is independent of the discretization parameters and prove some estimates. After defining the approximate solution in terms of these discrete points, we prove its convergence. These results include the results of a paper by author [14] (with Bahuguna). Next, we establish the existence of a unique local mild solution and its regularity under some additional condition with the help of the semigroup theory and the contraction mapping theorem under some different conditions. These results include the results of a paper by the author [15] (with Bahuguna).

In the closing Chapter 7, we consider some conclusions and provide some suggestions for further research work in this and related areas.

Relevant references are appended at the end.

Chapter 2

Sobolev Type Evolution Equations

One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics.

— Bertrand Russell

2.1 Introduction

In this chapter we shall study the approximations of the solution to the nonlinear Sobolev type evolution equation considered in a Hilbert space. Such type of equations arise in the analysis of a class of functional and neutral functional differential equations with unbounded delay. A functional differential equation is a differential equation in which the unknown function occurs with various different arguments. If x be unknown function of time t, then the delay differential equations (or differential equation with retarded argument) is an equation expressing some derivative of x at time t in terms of x (and its lower order derivatives if any) at t and at earlier instants.

Consider a class of the nonlinear Sobolev type evolutions equation of the form

$$\frac{d}{dt}(u(t) + g(t, u(t))) + Au(t) = f(t, u(t)), \quad t > 0, \quad u(0) = \phi, \tag{2.1.1}$$

in a separable Hilbert space $(H, \|.\|, (.,.))$, where the linear operator A satisfies the assumption (H1) stated in the next section so that $-A$ generates an analytic semigroup. The function f and g are appropriate continuous functions of their arguments in H.

The existence of a unique regular solution of (2.1.1) has been proved by Hernández [39] under the assumptions that $-A$ is the infinitesimal generator of analytic semigroup of bounded linear operators defined on a Banach space X and f and g are appropriate continuous functions on $[0, T] \times \Omega$ into X, where Ω is an open subset of X. Here, we are interested in establishing the convergence results for the Faedo-Galerkin approximations of solutions to (2.1.1).

The case of (2.1.1) in which $g \equiv 0$ has been extensively studied in literature, see, for instance, the books by Krein [64], Pazy [77], Goldstein [35] and the references cited in these books. The Faedo-Galerkin approximations of solution to the particular case of (2.1.1), where $g \equiv 0$ and $f(t, u) = M(u)$, has been considered by Miletta [72]. Our analysis is motivated by Miletta [72] and Bahuguna, Srivastava and Singh [16].

The study of (2.1.1), with linear g, has been initiated by Showalter [86, 87, 88, 89] with applications to some degenerate parabolic equations. Brill [25] has considered a pseudo-parabolic partial differential equation which appears in a variety of physical problems, for example, in thermodynamics [27], in the flow of fluid through fissured rocks [19], in the shear in second-order fluids [94] and in soil mechanics [93]. He has reformulated it as a semi-linear Sobolev evolution equation in a Banach space and established the existence results by using Schauder's fixed point principle. He has used the Sobolev imbedding theorems and the established abstract results for proving

some existence theorems for pseudo-parabolic partial differential equation under certain hypotheses.

The nonlinear Sobolev type evolution equations of the form (2.1.1) arise in the study of a partial neutral functional differential equation with unbounded delay which can be modelled in the form (cf. [40, 41])

$$\frac{d}{dt}(u(t) + G(t, u_t)) = Au(t) + F(t, u_t), \quad t > 0, \tag{2.1.2}$$

in a Banach space X, where A is the infinitesimal generator of an analytic semigroup in X, F and G are appropriate nonlinear functions from $[0, T] \times W$ into X and for any function $u \in C((-\infty, \infty), X)$, the history function $u_t \in C((-\infty, 0], X)$ of u, is given by $u_t(\theta) = u(t+\theta)$. The above equation is called an abstract neutral functional differential equation (ANFDE) with unbounded delay.

As a motivational example, for this class of equations, we consider the boundary value problem

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,x) + \int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,\eta,x)u(s,\eta)d\eta ds\right] = \frac{\partial^2}{\partial x^2}u(t,x) + a_0(x)u(t,x),\\
&\quad + \int_{-\infty}^{t} a(s-t)u(s,x)ds + a_1(t,x), \quad t \geq 0, \quad 0 \leq x \leq \pi,\\
&u(t,0) = u(t,\pi) = 0, \quad t \geq 0,\\
&u(\theta,x) = \phi(\theta,x), \quad \theta \leq 0, \quad 0 \leq x \leq \pi,
\end{aligned}$$

where the functions a_0, a, a_1, b_1 and ϕ satisfy appropriate conditions.

The organization of this chapter is as follows. In the second section, we state some preliminaries and the assumptions required to establish the convergence results. In the third section, we consider an integral equation associated with (2.1.1). A solution of this integral equation is defined as a mild solution of (2.1.1). With the help of the

associated integral equation and the projection operators, we consider a sequence of the approximate integral equations and establish the existence and uniqueness of a solution to each of the approximate integral equations. In the fourth section, we prove the convergence of the solutions of the approximate integral equations and show that the limiting function satisfies the associated integral equation. Further, we show in this section that the solution can be extended to the maximal interval of existence and it is unique. Finally, in the fifth section, we consider the Faedo-Galerkin approximations of solutions and prove some convergence results for such approximations.

2.2 Preliminaries and Assumptions

In this section we consider some preliminaries and assumptions essential for our purpose. The operator A assume to satisfy the following assumptions:

(H1) A is a closed, positive definite, self-adjoint, linear operator from the domain $D(A) \subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq, \cdots$$

and a corresponding complete orthonormal system of eigen functions $\{u_i\}$, i.e.,

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

These assumptions on A guarantee that $-A$ generates an analytic semigroup, denoted by e^{-tA}, $t \geq 0$.

Now, we mention some notions and preliminaries. It is well known that there exist constant $\tilde{M} \geq 1$ and a real number ω such that

$$\|e^{-tA}\| \leq \tilde{M} e^{\omega t}, \quad t \geq 0.$$

Since $-A$ generates the analytic semigroup e^{-tA}, $t \geq 0$, we may add cI to $-A$ for some constant c, if necessary, and in what follows we may assume without loss of generality that $\|e^{-tA}\|$ is uniformly bounded by M, i.e., $\|e^{-tA}\| \leq M$ and $0 \in \rho(A)$. In this case, it is possible to define the fractional power A^α for $0 < \alpha < 1$ as closed linear operator with domain $D(A^\alpha) \subseteq H$ (cf. Pazy [77], pp. 69-75 and p. 195). Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|,$$

defines a norm on $D(A^\alpha)$. Henceforth, we represent by X_α, the space $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$. In view of the facts mentioned above, we have the following result for an analytic semigroup e^{-tA}, $t \geq 0$ (cf. Pazy [77] pp. 195-196).

Lemma 2.2.1 *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup e^{-tA}, $t \geq 0$ with $\|e^{-tA}\| \leq M$ for $t \geq 0$ and $0 \in \rho(A)$. Then we have the following properties.*

(i) X_α is a Banach space for $0 \leq \alpha \leq 1$.

(ii) For $0 < \beta \leq \alpha$, the embedding $X_\alpha \hookrightarrow X_\beta$ is continuous.

(iii) A^α commutes with e^{-tA} and there exists a constant $C_\alpha > 0$ depending on α such that

$$\|A^\alpha e^{-tA}\| \leq C_\alpha t^{-\alpha}, \quad t > 0.$$

We assume the following assumptions on nonlinear maps f and g:

(H2) The nonlinear map f, defined from $[0,\infty) \times X_\alpha$ into H, is continuous and there exists a nondecreasing function F_R from $[0,\infty)$ into $[0,\infty)$ depending on $R > 0$ such that

$$\|f(t,u)\| \leq F_R(t),$$

$$\|f(t,u_1) - f(t,u_2)\| \leq F_R(t)\|u_1 - u_2\|_\alpha,$$

for all (t,u), (t,u_1) and (t,u_2) in $[0,\infty) \times B_R(X_\alpha,\phi)$, where

$$B_R(Z,z_0) = \{z \in Z \mid \|z - z_0\|_Z \leq R\},$$

for any Banach space Z with its norm $\|.\|_Z$.

(H3) There exist positive constants $0 < \alpha < \beta < 1$, L, R and $0 < \gamma \leq 1$ such that g is X_β- valued function and the map $A^\beta g$, defined from $[0,\infty) \times X_\alpha$ into H, satisfies

$$\|A^\beta g(t,u_1) - A^\beta g(s,u_2)\| \leq L\{|t-s|^\gamma + \|u_1 - u_2\|_\alpha\},$$

for all (t,u_1) and (s,u_2) in $[0,\infty) \times B_R(X_\alpha,\phi)$ with $L\|A^{\alpha-\beta}\| < 1$.

2.3 Approximate Integral Equations

We continue to use the notions and notations of the earlier section. The existence of solutions to (2.1.1) is closely associated with the existence of solutions to the integral equation

$$\begin{aligned} u(t) &= e^{-tA}(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t Ae^{-(t-s)A}g(s,u(s))ds \\ &\quad + \int_0^t e^{-(t-s)A}f(s,u(s))ds, \quad t \geq 0. \end{aligned} \tag{2.3.1}$$

In this section, we will consider an approximate integral equation associated to (2.3.1) and establish the existence and uniqueness of the solutions to the approximate integral equations. By a solution u to (2.3.1) on $[0,T]$, $0 < T < \infty$, we mean a function $u \in X_\alpha(T)$ for some $0 < \alpha < 1$ satisfying (2.3.1) on $[0,T]$, where $X_\alpha(T)$ is the Banach space $C([0,T],X_\alpha)$ of all continuous functions from $[0,T]$ into X_α endowed with the supremum norm

$$\|u\|_{X_\alpha(T)} = \sup_{0 \leq t \leq T} \|u(t)\|_\alpha.$$

By a solution u to (2.3.1) on $[0,\tilde{T})$, $0 < \tilde{T} \le \infty$, we mean a function u such that $u \in X_\alpha(T)$ for some $0 < \alpha < 1$ satisfying (2.3.1) on $[0, T]$ for every $0 < T < \tilde{T}$.

Let $0 < T_0 < \infty$ be arbitrarily fixed and let

$$B = \max_{0 \le t \le T_0} \|A^\beta g(t, \phi)\|.$$

We choose $0 < T \le T_0$ such that

$$\begin{aligned}
\|(e^{-tA} - I)A^\alpha(\phi + g(0, P^n\phi))\| &\le (1-\mu)\frac{R}{3}, \\
\|A^{\alpha-\beta}\|LT^\gamma + C_{1+\alpha-\beta}(L\tilde{R} + B)\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} &< (1-\mu)\frac{R}{6}, \\
C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} &< 1-\mu,
\end{aligned}$$

where $\mu = \|A^{\alpha-\beta}\|L$, $\tilde{R} = \sqrt{R^2 + \|\phi\|_\alpha^2}$ and C_α and $C_{1+\alpha-\beta}$ are the constants appear in Lemma 2.2.1.

Let H_n denote the finite dimensional subspace of the Hilbert space H spanned by $\{u_0, u_1, ...u_n\}$ and let $P^n : H \to H_n$ for $n = 0, 1, 2, \cdots$, be the corresponding projection operator. For each n, we define

$$f_n : [0, T] \times X_\alpha(T) \to H \quad \text{by} \quad f_n(t, u) = f(t, P^n u(t))$$

and

$$g_n : [0, T] \times X_\alpha(T) \to X_\beta(T) \quad \text{by} \quad g_n(t, u) = g(t, P^n u(t)).$$

We set $\tilde{\phi}(t) = \phi$ for $t \in [0, T]$ and for $n = 0, 1, 2, \cdots$, define a map S_n on $B_R(X_\alpha(T), \tilde{\phi})$ by

$$\begin{aligned}
(S_n u)(t) &= e^{-tA}(\phi + g_n(0, \tilde{\phi})) - g_n(t, u) + \int_0^t Ae^{-(t-s)A} g_n(s, u)ds \\
&+ \int_0^t e^{-(t-s)A} f_n(s, u)ds. \qquad (2.3.2)
\end{aligned}$$

Propostion 2.3.1 *Let (H1)-(H3) hold. Then there exists a unique function $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ such that $S_n u_n = u_n$ for each $n = 0, 1, 2, \ldots$; i.e., u_n satisfies the approximate integral equation*

$$\begin{aligned} u_n(t) &= e^{-tA}(\phi + g_n(0, \tilde{\phi})) - g_n(t, u_n) + \int_0^t Ae^{-(t-s)A} g_n(s, u_n) ds \\ &+ \int_0^t e^{-(t-s)A} f_n(s, u_n) ds. \end{aligned} \tag{2.3.3}$$

Proof. We claim that $S_n : B_R(X_\alpha(T), \tilde{\phi}) \to B_R(X_\alpha(T), \tilde{\phi})$. First, we show that the map $t \mapsto (S_n u)(t)$ is continuous from $[0, T]$ into X_α with respect to norm $\|.\|_\alpha$. For $t \in [0, T]$ and sufficiently small $h > 0$, we have

$$\begin{aligned} \|(S_n u)(t+h) &- (S_n u)(t)\|_\alpha \\ &\leq \|(e^{-hA} - I)A^\alpha e^{-tA}\|(\|\phi\| + \|g(0, P^n \phi)\|) \\ &+ \|A^{\alpha-\beta}\|\|A^\beta g_n(t+h, u) - A^\beta g_n(t, u)\| \\ &+ \int_0^t \|(e^{-hA} - I)A^{1+\alpha-\beta} e^{-(t-s)A}\|\|A^\beta g_n(s, u)\| ds \\ &+ \int_t^{t+h} \|e^{-(t+h-s)A} A^{1+\alpha-\beta}\|\|A^\beta g_n(s, u)\| ds \\ &+ \int_0^t \|(e^{-hA} - I)A^\alpha e^{-(t-s)A}\|\|f_n(s, u)\| ds \\ &+ \int_t^{t+h} \|e^{-(t+h-s)A} A^\alpha\|\|f_n(s, u)\| ds. \end{aligned} \tag{2.3.4}$$

Using (H3), we obtain

$$\begin{aligned} \|A^\beta g_n(t+h, u) - A^\beta g_n(t, u)\| &\leq L(h^\gamma + \|P^n u(t+h) - P^n u(t)\|_\alpha) \\ &\leq L(h^\gamma + \|u(t+h) - u(t)\|_\alpha) \end{aligned} \tag{2.3.5}$$

and

$$\int_t^{t+h} \|e^{-(t+h-s)A} A^{1+\alpha-\beta}\|\|A^\beta g_n(s, u)\| ds \leq \frac{(L\tilde{R} + B)C_{1+\alpha-\beta} h^{\beta-\alpha}}{\beta - \alpha}, \tag{2.3.6}$$

since

$$\begin{aligned}\|A^\beta g_n(s,u)\| &\leq \|A^\beta g_n(s,u) - A^\beta g(s,\phi)\| + \|A^\beta g(s,\phi)\| \\ &\leq L\|P^n u(s) - \phi\|_\alpha + B \ \leq \ L\tilde{R} + B \end{aligned} \tag{2.3.7}$$

and

$$\int_t^{t+h} \|e^{-(t+h-s)A}A^\alpha\| \|f_n(s,u)\| ds \ \leq \ \frac{C_\alpha F_{\tilde{R}}(T_0) h^{1-\alpha}}{1-\alpha}. \tag{2.3.8}$$

Part (d) of Theorem 1.3.19 implies that for $0 < \vartheta \leq 1$ and $x \in D(A^\vartheta)$,

$$\|(e^{-tA} - I)x\| \ \leq \ C'_\vartheta t^\vartheta \|x\|_\vartheta. \tag{2.3.9}$$

Let ϑ be a real number with $0 < \vartheta < \min\{1-\alpha, \beta-\alpha\}$, then $A^\alpha y \in D(A^\vartheta)$ for any $y \in D(A^{\alpha+\vartheta})$. For all $t, s \in [0,T]$, $t \geq s$ and $0 < h < 1$, we get the following inequalities

$$\|(e^{-hA} - I)A^\alpha e^{-tA}\| \ \leq \ C'_\vartheta h^\vartheta \|A^{\alpha+\vartheta} e^{-tA}\| \ \leq \ \frac{\tilde{C} h^\vartheta}{t^{\alpha+\vartheta}}, \tag{2.3.10}$$

$$\|(e^{-hA} - I)A^\alpha e^{-(t-s)A}\| \ \leq \ \frac{\tilde{C} h^\vartheta}{(t-s)^{\alpha+\vartheta}} \tag{2.3.11}$$

and

$$\|(e^{-hA} - I)A^{1+\alpha-\beta} e^{-(t-s)A}\| \ \leq \ \frac{\tilde{C} h^\vartheta}{t^{1+\alpha+\vartheta-\beta}}, \tag{2.3.12}$$

where $\tilde{C} = C'_\vartheta \max\{C_{\alpha+\vartheta}, C_{1+\alpha+\vartheta-\beta}\}$.

Using the estimates (2.3.7), (2.3.11) and (2.3.12), we get

$$\begin{aligned}&\int_0^t \|(e^{-hA} - I)A^{1+\alpha-\beta} e^{-(t-s)A}\| \|A^\beta g_n(s,u)\| ds \\ &\qquad \leq \ \tilde{C} h^\vartheta (L\tilde{R} + B) \frac{T_0^{\beta-(\alpha+\vartheta)}}{\beta - (\alpha+\vartheta)}\end{aligned} \tag{2.3.13}$$

and

$$\int_0^t \|(e^{-hA}-I)A^\alpha e^{-(t-s)A}\|\|f_n(s,u)\|ds \;\le\; \tilde{C}h^\vartheta F_{\tilde{R}}(T_0)\frac{T_0^{1-(\alpha+\vartheta)}}{1-(\alpha+\vartheta)}. \tag{2.3.14}$$

From the inequalities (2.3.4), (2.3.5), (2.3.6), (2.3.8), (2.3.10), (2.3.13) and (2.3.14), it follows that $(S_nu)(t)$ is continuous from $[0,T]$ into X_α with respect to the norm $\|.\|_\alpha$. Now, we show $S_nu \in B_R(X_\alpha(T),\tilde{\phi})$. Consider

$$\begin{aligned}
\|(S_nu)(t)-\phi\|_\alpha \;&\le\; \|(e^{-tA}-I)A^\alpha(\phi+g_n(0,\tilde{\phi}))\| \\
&\quad + \|A^{\alpha-\beta}\|\|A^\beta g_n(0,\tilde{\phi})-A^\beta g_n(t,u)\| \\
&\quad + \int_0^t \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\|A^\beta g_n(s,u)\|ds \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\|\|f_n(s,u)\|ds \\
&\le\; (1-\mu)\frac{R}{3} + \|A^{\alpha-\beta}\|L\{T^\gamma + \|u(t)-\phi\|_\alpha\} \\
&\quad + C_{1+\alpha-\beta}(L\tilde{R}+B)\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} \\
&\le\; (1-\mu)\frac{R}{3} + (1-\mu)\frac{R}{6} + \mu R \;\le\; R.
\end{aligned}$$

Taking supremum over $[0,T]$, we obtain

$$\|S_nu-\tilde{\phi}\|_{X_\alpha(T)} \;\le\; R.$$

Hence, S_n maps $B_R(X_\alpha(T),\tilde{\phi})$ into $B_R(X_\alpha(T),\tilde{\phi})$. Now, we show that S_n is a strict contraction on $B_R(X_\alpha(T),\tilde{\phi})$. For $u,v \in B_R(X_\alpha(T),\tilde{\phi})$, we have

$$\begin{aligned}
\|(S_nu)(t) \;&-\; (S_nv)(t)\|_\alpha \\
&\le\; \|A^{\alpha-\beta}\|\|A^\beta g_n(t,u)-A^\beta g_n(t,v)\|_\alpha \\
&\quad + \int_0^t \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\|A^\beta g_n(s,u)-A^\beta g_n(s,v)\|ds \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\|\|f_n(s,u)-f_n(s,v)\|ds.
\end{aligned} \tag{2.3.15}$$

Now,

$$\begin{aligned}\|A^\beta g_n(t,u) - A^\beta g_n(t,v)\| &\le L\|u(t)-v(t)\|_\alpha \\ &\le L\|u-v\|_{X_\alpha(T)}.\end{aligned} \tag{2.3.16}$$

Also, we have

$$\begin{aligned}\|f_n(s,u) - f_n(s,v)\| &\le F_{\tilde{R}}(T_0)\|u(s)-v(s)\|_\alpha \\ &\le F_{\tilde{R}}(T_0)\|u-v\|_{X_\alpha(T)}.\end{aligned} \tag{2.3.17}$$

Using (2.3.16) and (2.3.17) in (2.3.15), we get

$$\begin{aligned}&\|(S_n u)(t) - (S_n v)(t)\|_\alpha \\ &\le \left(\|A^{\alpha-\beta}\|L + C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha}\right)\|u-v\|_{X_\alpha(T)}.\end{aligned}$$

Taking supremum over $[0,T]$, we get

$$\begin{aligned}&\|S_n u - S_n v\|_{X_\alpha(T)} \\ &\le \left(\|A^{\alpha-\beta}\|L + C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha}\right)\|u-v\|_{X_\alpha(T)}.\end{aligned}$$

The above estimate and the definition of T imply that S_n is a strict contraction on $B_R(X_\alpha(T),\tilde{\phi})$. Hence there exists a unique $u_n \in B_R(X_\alpha(T),\tilde{\phi})$ such that $S_n u_n = u_n$. Clearly u_n satisfies (2.3.3). This completes the proof of the proposition.

Propostion 2.3.2 *Let (H1)-(H3) hold. If $\phi \in D(A^\alpha)$ for some $0<\alpha<\beta<1$, then $u_n(t) \in D(A^\vartheta)$ for all $t \in (0,T]$ where $0 \le \vartheta \le \beta < 1$. Furthermore, if $\phi \in D(A)$, then $u_n(t) \in D(A^\vartheta)$ for all $t \in [0,T]$ where $0 \le \vartheta \le \beta < 1$.*

Proof. From Proposition 2.3.1 we have the existence of a unique $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ satisfying (2.3.3). Part (a) of Theorem 1.3.19 implies that for $t > 0$ and $0 \le \vartheta < 1$, $e^{-tA} : H \to D(A^\vartheta)$ and for $0 \le \vartheta \le \beta < 1$, $D(A^\beta) \subseteq D(A^\vartheta)$. (H3) implies that the map $t \mapsto A^\beta g(t, u_n(t))$ is Hölder continuous on $[0, T]$. It follows that (cf. Theorem 4.3.2 in [77])

$$\int_0^t e^{-(t-s)A} A^\beta g_n(s, u_n) ds \in D(A).$$

Also from Theorem 1.3.9 we have $e^{-tA}x \in D(A)$ if $x \in D(A)$. The required result follows from these facts and the fact that $D(A) \subseteq D(A^\vartheta)$ for $0 \le \vartheta \le 1$.

Propostion 2.3.3 *Let (H1)-(H3) hold. Then for any $\phi \in D(A^\alpha)$, $0 < \alpha < \beta < 1$ and for any $t_0 \in (0, T]$, there exists a constant U_{t_0}, independent of n, such that*

$$\|u_n(t)\|_\vartheta \le U_{t_0}, \quad 0 \le \vartheta < \beta < 1, \quad t_0 \le t \le T.$$

Moreover, if $\phi \in D(A)$, then there exists a constant U_0, independent of n, such that

$$\|u_n(t)\|_\vartheta \le U_0, \quad 0 \le \vartheta < \beta < 1, \quad 0 \le t \le T.$$

Proof. Applying A^ϑ on both the sides of (2.3.3) and using (iii) of Lemma 2.2.1, for $t_0 \le t \le T$, we have

$$\begin{aligned}
\|u_n(t)\|_\vartheta &\le \|e^{-tA}A^\vartheta(\phi + g_n(0, \tilde{\phi}))\| + \|A^{\vartheta-\beta}\|\|A^\beta g_n(t, u_n)\| \\
&\quad + \int_0^t \|A^{1+\vartheta-\beta} e^{-(t-s)A}\|\|A^\beta g_n(s, u_n)\| ds \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\vartheta\|\|f_n(s, u_n)\| ds \\
&\le C_\vartheta t_0^{-\vartheta}(\|\phi\| + \|g_n(0, \tilde{\phi}\|) + \|A^{\vartheta-\beta}\|(L\tilde{R} + B) \\
&\quad + C_{1+\vartheta-\beta}(L\tilde{R} + B)\frac{T^{\beta-\vartheta}}{\beta - \vartheta} + C_\vartheta F_{\tilde{R}}(T_0)\frac{T^{1-\vartheta}}{1-\vartheta} \le U_{t_0}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\|A^{\beta}g_n(t,u_n) - A^{\beta}g_m(t,u_m)\| \\
&\quad\leq \|A^{\beta}g_n(t,u_n) - A^{\beta}g_n(t,u_m)\| + \|A^{\beta}g_n(t,u_m) - A^{\beta}g_m(t,u_m)\| \\
&\quad\leq L[\|u_n(t) - u_m(t)\|_{\alpha} + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^{\vartheta}u_m(t)\|].
\end{aligned}$$

Now, for $0 < t_0' < t_0$, we may write

$$\begin{aligned}
&\|u_n(t) - u_m(t)\|_{\alpha} \\
&\leq \|e^{-tA}A^{\alpha}(g_n(0,\tilde{\phi}) - g_m(0,\tilde{\phi}))\| + \|A^{\alpha-\beta}\|\|A^{\beta}g_n(t,u_n) - A^{\beta}g_m(t,u_m)\| \\
&\quad + \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right)\|A^{1+\alpha-\beta}e^{-(t-s)A}\|\|A^{\beta}g_n(s,u_n) - A^{\beta}g_m(s,u_m)\|ds \\
&\quad + \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right)\|A^{\alpha}e^{-(t-s)A}\|\|f_n(s,u_n) - f_m(s,u_m)\|ds.
\end{aligned}$$

We estimate the first term as

$$\begin{aligned}
\|e^{-tA}A^{\alpha}(g_n(0,\tilde{\phi}) &- g_m(0,\tilde{\phi}))\| \\
&\leq M\|A^{\alpha-\beta}\|\|A^{\beta}g(0,P^n\phi) - A^{\beta}g(0,P^m\phi)\| \\
&\leq M\|A^{\alpha-\beta}\|L\|(P^n - P^m)\phi\|_{\alpha}.
\end{aligned}$$

The first and the third integrals are estimated as

$$\begin{aligned}
&\int_0^{t_0'}\|A^{1+\alpha-\beta}e^{-(t-s)A}\|\|A^{\beta}g_n(s,u_n) - A^{\beta}g_m(s,u_m)\|ds \\
&\qquad\qquad\leq 2C_{1+\alpha-\beta}(L\tilde{R} + B)(t_0 - t_0')^{-(1+\alpha-\beta)}t_0',
\end{aligned}$$

$$\begin{aligned}
&\int_0^{t_0'}\|A^{\alpha}e^{-(t-s)A}\|\|f_n(s,u_n) - f_m(s,u_m)\|ds \\
&\qquad\qquad\leq 2C_{\alpha}F_{\tilde{R}}(T_0)(t_0 - t_0')^{-\alpha}t_0'.
\end{aligned}$$

For the second and the fourth integrals, we have

$$\begin{aligned}
&\int_{t_0'}^{t} \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\|A^{\beta}g_n(s,u_n) - A^{\beta}g_m(s,u_m)\|ds \\
&\quad\le C_{1+\alpha-\beta}L\int_{t_0'}^{t}(t-s)^{-(1+\alpha-\beta)}\left(\|u_n(s)-u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^{\vartheta}u_m(s)\|\right)ds \\
&\quad\le C_{1+\alpha-\beta}L\left(\frac{U_{t_0'}T^{\beta-\alpha}}{\lambda_m^{\vartheta-\alpha}(\beta-\alpha)} + \int_{t_0'}^{t}(t-s)^{-(1+\alpha-\beta)}\|u_n(s)-u_m(s)\|_\alpha ds\right),
\end{aligned}$$

$$\begin{aligned}
&\int_{t_0'}^{t} \|A^{\alpha}e^{-(t-s)A}\|\|f_n(s,u_n) - f_m(s,u_m)\|ds \\
&\quad\le C_\alpha F_{\tilde{R}}(T_0)\int_{t_0'}^{t}(t-s)^{-\alpha}\left(\|u_n(s)-u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^{\vartheta}u_m(s)\|\right)ds \\
&\quad\le C_\alpha F_{\tilde{R}}(T_0)\left(\frac{U_{t_0'}T^{1-\alpha}}{\lambda_m^{\vartheta-\alpha}(1-\alpha)} + \int_{t_0'}^{t}(t-s)^{-\alpha}\|u_n(s)-u_m(s)\|_\alpha ds\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|u_n(t) &- u_m(t)\|_\alpha \\
&\le ML\|A^{\alpha-\beta}\|\|(P^n-P^m)\phi\|_\alpha \\
&\quad + L\|A^{\alpha-\beta}\|\left(\|u_n(t)-u_m(t)\|_\alpha + \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\right) \\
&\quad + 2\left(\frac{C_{1+\alpha-\beta}(L\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0-t_0')^{\alpha}}\right)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&\quad + \int_{t_0'}^{t}\left(\frac{C_\alpha F_{\tilde{R}}(T_0)}{(t-s)^{\alpha}} + \frac{C_{1+\alpha-\beta}L}{(t-s)^{1+\alpha-\beta}}\right)\|u_n(s)-u_m(s)\|_\alpha ds,
\end{aligned}$$

where

$$C_{\alpha,\beta} = C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} + C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha}.$$

Since $\|A^{\alpha-\beta}\|L < 1$, we have

$$\begin{aligned}
\|u_n(t) &- u_m(t)\|_\alpha \\
&\le \frac{1}{(1-\|A^{\alpha-\beta}\|L)} \Bigg\{ M\|(P^n - P^m)\phi\|_\alpha + L\|A^{\alpha-\beta}\| \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&+ 2\left(\frac{C_{1+\alpha-\beta}(L\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0-t_0')^\alpha} \right) t_0' + C_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&+ \int_{t_0'}^{t} \left(\frac{C_\alpha F_{\tilde{R}}(T_0)}{(t-s)^\alpha} + \frac{C_{1+\alpha-\beta}L}{(t-s)^{1+\alpha-\beta}} \right) \|u_n(s) - u_m(s)\|_\alpha ds \Bigg\}. \qquad (2.4.1)
\end{aligned}$$

Consider

$$\begin{aligned}
\left(\frac{C_\alpha F_{\tilde{R}}(T_0)}{(t-s)^\alpha} + \frac{C_{1+\alpha-\beta}L}{(t-s)^{1+\alpha-\beta}} \right) &\le N_1 \left(\frac{1}{(t-s)^\alpha} + \frac{1}{(t-s)^{1+\alpha-\beta}} \right) \\
&\le N_1(T^{1-\beta}+1) \frac{1}{(t-s)^{1+\alpha-\beta}},
\end{aligned}$$

where

$$N_1 = \max\{C_\alpha F_{\tilde{R}}(T_0), C_{1+\alpha-\beta}L\}.$$

Using this estimate in (2.4.1), we get

$$\begin{aligned}
\|u_n(t) &- u_m(t)\|_\alpha \\
&\le \frac{1}{(1-\|A^{\alpha-\beta}\|L)} \Bigg\{ M\|(P^n - P^m)\phi\|_\alpha + \|A^{\alpha-\beta}\|L \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&+ 2\left(\frac{C_{1+\alpha-\beta}(L\tilde{R}+B)}{(t_0-t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0-t_0')^\alpha} \right) t_0' + C_{\alpha,\beta} \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}} \\
&+ N(T^{1-\beta}+1) \int_{t_0'}^{t} \frac{1}{(t-s)^{1+\alpha-\beta}} \|u_n(s) - u_m(s)\|_\alpha ds \Bigg\}.
\end{aligned}$$

Hence from lemma 1.3.27, there exists a constant K such that

$$\begin{aligned} &\|u_n(t) - u_m(t)\|_\alpha \\ \leq\ & \frac{1}{(1 - \|A^{\alpha-\beta}\|L)} \left\{ M\|(P^n - P^m)\phi\|_\alpha + (\|A^{\alpha-\beta}\|L + C_{\alpha,\beta}) \frac{U_{t'_0}}{\lambda_m^{\vartheta-\alpha}} \right. \\ & \left. + 2\left(\frac{C_{1+\alpha-\beta}(L\tilde{R} + B)}{(t_0 - t'_0)^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0 - t'_0)^\alpha} \right) t'_0 \right\} K. \end{aligned}$$

Taking supremum over $[t_0, T]$ and letting $m \to \infty$, we obtain

$$\begin{aligned} &\lim_{m\to\infty} \sup_{\{n\geq m, t\in[t_0,T]\}} \|u_n(t) - u_m(t)\|_\alpha \\ \leq\ & \frac{2}{(1 - \|A^{\alpha-\beta}\|L)} \left(\frac{C_{1+\alpha-\beta}(L\tilde{R} + B)}{(t_0 - t'_0)^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0 - t'_0)^\alpha} \right) K t'_0. \end{aligned}$$

As t'_0 is arbitrary, the right hand side may be made as small as desired by taking t'_0 sufficiently small. This completes the proof of the proposition.

Corollary 2.4.2 *If $\phi \in D(A)$, then*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ 0\leq t\leq T\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof. Propositions 2.3.2 and 2.3.3 imply that in the proof of Proposition 2.4.1 we may take $t_0 = 0$.

For the convergence of the solution $u_n(t)$ of approximate integral equation (2.3.3) we have the following result.

Theorem 2.4.3 *Let (H1)-(H3) hold and let $\phi \in D(A^\alpha)$. Then there exists a unique function $u \in X_\alpha(T)$ such that $u_n \to u$ as $n \to \infty$ in $X_\alpha(T)$ and u satisfies (2.3.1) on $[0, T]$. Furthermore, u can be extended to the maximal interval of existence $[0, t_{max})$, $0 < t_{max} \leq \infty$ satisfying (2.3.1) on $[0, t_{max})$ and u is a unique solution to (2.3.1) on $[0, t_{max})$.*

Proof. We first assume that $\phi \in D(A)$. Corollary 2.4.2 implies that there exists $u \in X_\alpha(T)$ such that u_n converges to u in X_α. Since $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ for each n, u is also in $B_R(X_\alpha(T), \tilde{\phi})$. Further, we have

$$\|g_n(0, \tilde{\phi}) - g(0, \phi)\| \leq \|g(0, P^n\phi) - g(0, \phi)\| \leq L\|(P^n - I)\phi\|_\alpha \tag{2.4.2}$$

and the right hand side of (2.4.2) tends to zero as $n \to \infty$. Also,

$$\begin{aligned} \|f_n(t, u_n) &- f(t, u(t))\| \\ &\leq \|f_n(t, u_n) - f_n(t, u)\| + \|f_n(t, u) - f(t, u(t))\| \\ &\leq F_{\tilde{R}}(T_0)(\|u_n(t) - u(t)\|_\alpha + \|(P^n - I)u(t)\|_\alpha). \end{aligned}$$

Taking supremum over $[0, T]$, we get

$$\begin{aligned} \sup_{0 \leq t \leq T} \|f_n(t, u_n) &- f(t, u(t))\| \\ &\leq F_{\tilde{R}}(T_0)(\|u_n - u\|_{X_\alpha(T)} + \|(P^n - I)u\|_{X_\alpha(T)}). \end{aligned} \tag{2.4.3}$$

The right hand side of (2.4.3) tends to zero as $n \to \infty$. From (H3), we get

$$\begin{aligned} \|A^\beta g_n(t, u_n) &- A^\beta g(t, u(t))\| \\ &\leq \|A^\beta g_n(t, u_n) - A^\beta g_n(t, u)\| + \|A^\beta g_n(t, u) - A^\beta g(t, u(t))\| \\ &\leq L(\|u_n(t) - u(t)\|_\alpha + \|(P^n - I)u(t)\|_\alpha). \end{aligned}$$

Taking supremum over $[0, T]$, we get

$$\begin{aligned} &\sup_{0 \leq t \leq T} \|A^\beta g_n(t, u_n) - A^\beta g(t, u(t))\| \\ &\leq L(\|u_n - u\|_{X_\alpha(T)} + \|(P^n - I)u\|_{X_\alpha(T)}) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{2.4.4}$$

Using (2.4.2), (2.4.3), (2.4.4) and the bounded convergence theorem in (2.3.3), we get

$$\begin{aligned} u(t) &= e^{-tA}(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t Ae^{-(t-s)A}g(s,u(s))ds \\ &+ \int_0^t e^{-(t-s)A}f(s,u(s))ds. \end{aligned} \tag{2.4.5}$$

Thus, for $\phi \in D(A)$, there exists a unique function $u \in X_\alpha(T)$ such that $u_n \to u$ as $n \to \infty$ and satisfies integral equation (2.3.1) on $[0,T]$.

Now, let us assume that $\phi \in D(A^\alpha)$. Since, for $0 < t \leq T$, $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ as $n \to \infty$ and $u_n(0) = u(0) = \phi$ for all n, we have, for $0 \leq t \leq T$, $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ in H as $n \to \infty$. Since $u_n \in B_R(X_\alpha(T), \tilde{\phi})$, it follows that $u \in B_R(X_\alpha(T), \tilde{\phi})$ and for any $0 < t_0 \leq T$,

$$\lim_{n\to\infty} \sup_{\{t_0 \leq t \leq T\}} \|u_n(t) - u(t)\|_\alpha = 0.$$

Also,

$$\begin{aligned} &\sup_{t_0 \leq t \leq T} \|f_n(t,u_n) - f(t,u(t))\| \\ &\leq F_{\tilde{R}}(T_0)(\|u_n - u\|_{X_\alpha(T)} + \|(P^n - I)u\|_{X_\alpha(T)}) \to 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

and

$$\begin{aligned} &\sup_{t_0 \leq t \leq T} \|A^\beta g_n(t,u_n) - A^\beta g(t,u(t))\| \\ &\leq L(\|u_n - u\|_{X_\alpha(T)} + \|(P^n - I)u\|_{X_\alpha(T)}) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now, for $0 < t_0 < t$, we may rewrite (2.3.3) as

$$\begin{aligned} u_n(t) &= e^{-tA}(\phi + g_n(0,\tilde{\phi})) - g_n(t,u_n) + \left(\int_0^{t_0} + \int_{t_0}^t\right) Ae^{-(t-s)A}g_n(s,u_n)ds \\ &+ \left(\int_0^{t_0} + \int_{t_0}^t\right) e^{-(t-s)A}f_n(s,u_n)ds. \end{aligned}$$

The first and third integrals are estimated as

$$\begin{aligned}\left\|\int_0^{t_0} Ae^{-(t-s)A}g_n(s,u)ds\right\| &\leq \int_0^{t_0} \|A^{1-\beta}e^{-(t-s)A}\|\|A^{\beta}g_n(s,u_n)\|ds\\ &\leq C_{1-\beta}(L\tilde{R}+B)T^{1-\beta}t_0,\end{aligned}$$

$$\left\|\int_0^{t_0} e^{-(t-s)A}f_n(s,u_n)ds\right\| \leq MF_{\tilde{R}}(T_0)t_0.$$

Thus, we have

$$\begin{aligned}&\left\|u_n(t) - e^{-tA}(\phi + g_n(0,\tilde{\phi})) + g_n(t,u_n) - \int_{t_0}^{t} Ae^{-(t-s)A}g_n(s,u_n)ds\right.\\ &\left.- \int_{t_0}^{t} e^{-(t-s)A}f_n(s,u_n)ds\right\| \leq (C_{1-\beta}(L\tilde{R}+B)T^{1-\beta} + MF_{\tilde{R}}(T_0))t_0.\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned}&\left\|u(t) - e^{-tA}(\phi + g(0,\phi)) + g(t,u(t)) - \int_{t_0}^{t} Ae^{-(t-s)A}g(s,u(s))ds\right.\\ &\left.- \int_{t_0}^{t} e^{-(t-s)A}f(s,u(s))ds\right\| \leq (C_{1-\beta}(L\tilde{R}+B)T^{1-\beta} + MF_{\tilde{R}}(T_0))t_0.\end{aligned}$$

Since $0 < t_0 \leq T$ is arbitrary, we obtain that u satisfies the integral equation (2.3.1).

If u satisfies (2.3.1) on $[0,T]$, then we show that, u can be extended further. Since $0 < T_0 < \infty$, was arbitrary, we assume that $0 < T < T_0$. We consider the equation

$$\begin{aligned}\frac{d}{dt}(w(t) + G(t,w(t))) + Aw(t) &= F(t,w(t)), \quad 0 \leq t \leq T_0 < \infty,\\ w(0) &= u(T),\end{aligned}$$

where $F : [0, T_0 - T] \times X_\alpha \longrightarrow H$ is defined by

$$F(t,x) = f(t+T,x),$$

and $G : [0, T_0 - T] \times X_\alpha \to X_\beta$ is defined by

$$G(t, x) = g(t + T, x),$$

for $(t, x) \in [0, T_0 - T] \times X_\alpha$. We note that F and G satisfy (H2) and (H3), respectively, where T_0 is replaced by $T_0 - T$. Hence, there exists a unique $w \in C([0, T_1], X_\alpha)$, for some $0 < T_1 < T_0 - T$, satisfying the integral equation

$$\begin{aligned} w(t) &= e^{-tA}(u(T) + G(0, u(T))) - G(t, w(t)) \\ &\quad + \int_0^t Ae^{-(t-s)A}G(s, w(s))ds + \int_0^t e^{-(t-s)A}F(s, w(s))ds, \quad 0 \le t \le T_1. \end{aligned}$$

Now, we define

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \le t \le T, \\ w(t - T), & T \le t \le T_1 + T. \end{cases}$$

Then, $\tilde{u}$ satisfies the integral equation

$$\begin{aligned} \tilde{u}(t) &= e^{-tA}(\phi + g(0, \phi)) - g(t, \tilde{u}(t)) + \int_0^t Ae^{-(t-s)A}g(s, \tilde{u}(s))ds \\ &\quad + \int_0^t e^{-(t-s)A}f(s, \tilde{u}(s))ds, \quad 0 \le t \le T_1 + T. \end{aligned} \tag{2.4.6}$$

To see this, we need to verify (2.4.6) only on $[T, T_1 + T]$. For $t \in [T, T_1 + T]$,

$$\begin{aligned} \tilde{u}(t) &= w(t - T) \\ &= e^{-(t-T)A}(u(T) + G(0, u(T))) - G(t - T, w(t - T)) \\ &\quad + \int_0^{t-T} Ae^{-(t-T-s)A}G(s, w(s))ds + \int_0^{t-T} e^{-(t-T-s)A}F(s, w(s))ds. \end{aligned}$$

Putting $T + s = \eta$, we get

$$\begin{aligned}
\tilde{u}(t) &= e^{-(t-T)A}(\{e^{-TA}(\phi + g(0,\phi)) - g(T,u(T)) \\
&\quad + \int_0^T Ae^{-(T-s)A}g(s,u(s))ds + \int_0^T e^{-(T-s)A}f(s,u(s))ds\} \\
&\quad + G(0,u(T))) - G(t-T,w(t-T)) \\
&\quad + \int_T^t Ae^{-(t-\eta)A}G(\eta - T, w(\eta - T))d\eta \\
&\quad + \int_T^t e^{-(t-\eta)A}F(\eta - T, w(\eta - T))ds \\
&= e^{-tA}(\phi + g(0,\phi)) - g(t,w(t-T)) + \int_0^T Ae^{-(t-s)A}g(s,u(s))ds \\
&\quad + \int_T^t Ae^{-(t-s)A}g(s,w(s-T))ds + \int_0^T e^{-(t-s)A}f(s,u(s))ds \\
&\quad + \int_T^t e^{-(t-s)A}f(s,w(s-T))ds,
\end{aligned}$$

as $G(0,u(T)) = g(T,u(T))$, $G(t-T,w(t-T)) = g(t,w(t-T))$ and $F(t-T,w(t-T)) = f(t,w(t-T))$. Hence, we have

$$\begin{aligned}
\tilde{u}(t) &= e^{-tA}(\phi + g(0,\phi)) - g(t,\tilde{u}(t)) + \int_0^t Ae^{-(t-s)A}g(s,\tilde{u}(s))ds \\
&\quad + \int_0^t e^{-(t-s)A}f(s,\tilde{u}(s))ds,
\end{aligned}$$

for $t \in [0, T_1+T]$. Thus, we see $\tilde{u}(t)$ satisfy (2.3.1) on $[0, T_1+T]$. Hence, we may extend $u(t)$ to maximal interval $[0, t_{max})$ satisfying (2.3.1) on $[0, t_{max})$ with $0 < t_{max} \leq \infty$.

Now, we show the uniqueness of solutions to (2.3.1). Let u_1 and u_2 be two solutions to (2.3.1) on some interval $[0, T]$, where T be any number such that $0 < T < t_{max}$. Then,

for $0 \le t \le T$, we have

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|_\alpha \\
\le\ & \|A^{\alpha-\beta}\| \|A^\beta g(t, u_1(t)) - A^\beta g(t, u_2(t))\| \\
&+ \int_0^t \|A^{1+\alpha-\beta} e^{-(t-s)A}\| \|A^\beta g(s, u_1(s)) - A^\beta g(s, u_2(s))\| ds \\
&+ \int_0^t \|e^{-(t-s)A} A^\alpha\| \|f(s, u_1(s)) - f(s, u_2(s))\| ds \\
\le\ & \|A^{\alpha-\beta}\| L \|u_1(t) - u_2(t)\|_\alpha \\
&+ C_{1+\alpha-\beta} L \int_0^t (t-s)^{-(1+\alpha-\beta)} \|u_1(s) - u_2(s)\|_\alpha ds \\
&+ C_\alpha F_{\tilde{R}}(t_{max}) \int_0^t (t-s)^{-\alpha} \|u_1(s) - u_2(s)\|_\alpha ds.
\end{aligned}$$

Since, $\|A^{\alpha-\beta}\| L < 1$, we have

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|_\alpha \\
\le\ & \frac{1}{(1 - \|A^{\alpha-\beta}\| L)} \int_0^t \left(\frac{C_{1+\alpha-\beta} L}{(t-s)^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(t_{max})}{(t-s)^\alpha} \right) \|u_1(s) - u_2(s)\|_\alpha ds \\
\le\ & \frac{N_2 (T^{1-\beta} + 1)}{(1 - \|A^{\alpha-\beta}\| L)} \int_0^t \frac{1}{(t-s)^{1+\alpha-\beta}} \|u_1(s) - u_2(s)\|_\alpha ds,
\end{aligned}$$

where

$$N_2 = \max\{C_\alpha F_{\tilde{R}}(t_{max}), C_{1+\alpha-\beta} L\}.$$

Using Lemma 1.3.27, we get

$$\|u_1(t) - u_2(t)\|_\alpha = 0$$

for all $0 \le t \le T$. From the fact that

$$\|u_1(t) - u_2(t)\| \le \frac{1}{\lambda_0^\alpha} \|u_1(t) - u_2(t)\|_\alpha,$$

it follows that $u_1 = u_2$ on $[0, T]$. Since $0 < T < t_{max}$ was arbitrary, we have $u_1 = u_2$ on $[0, t_{max})$. This completes the proof of the theorem.

2.5 Faedo-Galerkin Approximations

For any $0 < T < t_{max}$, we have a unique $u \in X_\alpha(T)$ satisfying the integral equation

$$\begin{aligned} u(t) &= e^{-tA}(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t Ae^{-(t-s)A}g(s,u(s))ds \\ &\quad + \int_0^t e^{-(t-s)A}f(s,u(s))ds. \end{aligned} \tag{2.5.1}$$

Also, we have a unique solution $u_n \in X_\alpha(T)$ of the approximate integral equation

$$\begin{aligned} u_n(t) &= e^{-tA}(\phi + g_n(0,\tilde{\phi})) - g_n(t,u_n) + \int_0^t Ae^{-(t-s)A}g_n(s,u_n)ds \\ &\quad + \int_0^t e^{-(t-s)A}f_n(s,u_n)ds. \end{aligned} \tag{2.5.2}$$

If we project (2.5.2) onto H_n, we get the Faedo-Galerkin approximation $\hat{u}_n(t) = P^n u_n(t)$ satisfying

$$\begin{aligned} \hat{u}_n(t) &= e^{-tA}(P^n\phi + P^n g(0,P^n\phi)) - P^n g(t,\hat{u}_n(t)) \\ &\quad + \int_0^t Ae^{-(t-s)A}P^n g(s,\hat{u}_n(s))ds + \int_0^t e^{-(t-s)A}P^n f(s,\hat{u}_n(s))ds \end{aligned} \tag{2.5.3}$$

The solution u of (2.5.1) and $\hat{u}_n$ of (2.5.3), have the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)u_i, \quad \alpha_i(t) = (u(t),u_i), \quad i = 0,1,\ldots; \tag{2.5.4}$$

$$\hat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)u_i, \quad \alpha_i^n(t) = (\hat{u}_n(t),u_i), \quad i = 0,1,\ldots n. \tag{2.5.5}$$

Using (2.5.5) in (2.5.3), we obtain the following system of first order ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\left(\alpha_i^n(t) + G_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t))\right) + \lambda_i \alpha_i^n(t) &= F_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t)), \\ \alpha_i^n(0) &= \phi_i, \end{aligned} \tag{2.5.6}$$

where

$$G_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t)) = \left(g(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i), u_i \right),$$

$$F_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t)) = \left(f(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i), u_i \right)$$

and $\phi_i = (\phi, u_i)$ for $i = 1, 2, ...n$.

The system (2.5.6) determines the $\alpha_i^n(t)$'s. Now, we shall show the convergence of $\alpha_i^n(t) \to \alpha_i(t)$. It can easily be checked that

$$\begin{aligned} A^\alpha[u(t) - \hat{u}(t)] &= A^\alpha \left[\sum_{i=0}^{\infty} (\alpha_i(t) - \alpha_i^n(t)) u_i \right] \\ &= \sum_{i=0}^{\infty} \lambda_i^\alpha (\alpha_i(t) - \alpha_i^n(t)) u_i. \end{aligned}$$

Thus, we have

$$\|A^\alpha[u(t) - \hat{u}(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} (\alpha_i(t) - \alpha_i^n(t))^2.$$

We have the following convergence Theorem.

Theorem 2.5.1 *Let H1)-(H3) hold. Then we have the following .*

(a) If $\phi \in D(A^\alpha)$, then for any $0 < t_0 \leq T$,

$$\lim_{n \to \infty} \sup_{t_0 \leq t \leq T} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha} (\alpha_i(t) - \alpha_i^n(t))^2 \right] = 0.$$

(b) If $\phi \in D(A)$, then

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left[\sum_{i=0}^{n} \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2\right] = 0.$$

The assertion of Theorem 2.5.1 follows from the facts mentioned above and the following proposition.

Propostion 2.5.2 *Let (H1)-(H3) hold and let T be any number such that $0 < T < t_{max}$, then we have the following.*

(a) If $\phi \in D(A^\alpha)$, then for any $0 < t_0 \le T$,

$$\lim_{m\to\infty} \sup_{\{n\ge m,\ t_0\le t\le T\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

(b) If $\phi \in D(A)$, then

$$\lim_{m\to\infty} \sup_{\{n\ge m,\ 0\le t\le T\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

Proof. For $n \ge m$, we have

$$\begin{aligned}
\|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| &= \|A^\alpha[P^n u_n(t) - P^m u_m(t)]\| \\
&\le \|P^n[u_n(t) - u_m(t)]\|_\alpha + \|(P^n - P^m)u_m\|_\alpha \\
&\le \|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m\|.
\end{aligned}$$

If $\phi \in D(A^\alpha)$, then the result in (a) follows directly from Proposition 2.4.1. If $\phi \in D(A)$, (b) follows from Corollary 2.4.2.

Chapter 3

Second Order semi-linear Evolution Equations

An approximate answer to the right problem is worth a good deal more than an exact answer to an approximate problem.

— John Tukey

3.1 Introduction

In this chapter, we shall study the approximation of the solution to a second order semi-linear evolution equation. We first show that a strongly damped semi-linear wave equation may be written as a second order semi-linear evolution equation.

Let Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and let

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$$

be a symmetric second order strongly elliptic differential operator in Ω. We are concerned with the following initial-boundary value problem for a strongly damped semi-

linear wave equation,

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(x,t) + (aL + bI)\left(\frac{\partial u}{\partial t}\right)(x,t) + (cL + dI)u(x,t) \\
&\qquad = h(x,t,u(x,t),\frac{\partial u}{\partial t}(x,t)), \quad (x,t) \in \Omega \times (0,T), \qquad (3.1.1) \\
&u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x \in \Omega,
\end{aligned}$$

with the homogeneous Dirichlet boundary conditions, where $a > 0$, b, c and d are constants and h is a smooth function of its arguments.

Under suitable conditions on the data h, ϕ and ψ, we reformulate (3.1.1) as the following initial value problem for the abstract second order semi-linear equation,

$$\begin{aligned}
&\frac{d^2 u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) + Bu(t) = f(t,u(t),\frac{du}{dt}(t)), \quad t > 0, \\
&u(0) = x_0, \quad \frac{du}{dt}(0) = x_1, \qquad (3.1.2)
\end{aligned}$$

in the separable Hilbert space $H = L^2(\Omega)$ where the linear operator A with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ is given by

$$Au = aLu, \quad u \in D(A),$$

and the operator B is such that $D(B) = D(A)$ with $Bu = c_1 A + c_2 I$ for some constant c_1 and c_2. The function f is defined from $[0,\infty) \times H \times H$ into H given by

$$f(t,u,v) = h(t,u,v) - bv.$$

D. Henry [38] has considered the coupled system

$$\frac{dv}{dt} + Av = f(t,u,v), \quad \frac{du}{dt} = g(t,u,v), \qquad (3.1.3)$$

and proved the existence of an invariant manifold (collection of solutions) and established the stability theory for this system under certain assumptions on A, f and g. We may write (3.1.2) as a system of the first order equations like (3.1.3), where $g(t,u,v) = v$.

The problem (3.1.1), for the particular case $L = -\Delta$, the n-dimensional Laplacian, is dealt with by Duvaut and Lions [31] and Glowinski, Lions and Tremolieres [34] in the context of some viscoelastic materials using the method of variational inequalities.

Sandefur [84] has considered the problem (3.1.2), for a special case where the operators A and B are such that the equation can be written in the following factorized form,

$$\begin{aligned} &\frac{d^2u}{dt^2} - (A_1 + A_2)\left(\frac{du}{dt}\right) + A_1A_2u = f(t,u), \quad t \in (0,T], \\ &u(0) = x_0, \quad \frac{du}{dt}(0) = x_1. \end{aligned} \tag{3.1.4}$$

Using the theory of semigroups together with the method of successive approximations, Sendefur [84] applied the results to various physical problems, for instance, damped and strongly damped semi-linear equations, the telegraph equation and the equation of motion for a thin panel.

Aviles and Sandefur [3] have modified the techniques of Sandefur [84] by changing the order of integration in the definition of the mild solution to (3.1.4) and applied the results to the Klein-Gordon equation, the von Karman equation and the vibrating beam equation.

Webb [95] and Ang and Dinh [2] have considered the following initial boundary value problem,

$$\begin{aligned} &u_{tt} - \lambda\Delta u_t + f(u) = 0, \quad (x,t) \in \Omega \times (0,T), \quad \lambda > 0, \\ &u(x,t) = 0, \quad .(x,t) \in \partial\Omega \times [0,T), \\ &u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \end{aligned} \tag{3.1.5}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$. For $1 \leq$

$N \leq 2$, (3.1.5) governs the motion of a linear Kelvin solid (a bar if $N = 1$, a plate if $N = 2$) subject to nonlinear elastic constraints. Webb [95] has studied the existence and asymptotic behavior of solutions to (3.1.5) for $N = 1, 2, 3$. Ang and Dinh [2] have generalized the results of Webb [95] for (3.1.5) using the method of successive linearizations.

Engler, Neubrander and Sandefur [32] have used the theory of semigroups to prove the existence and uniqueness of mild solutions of (3.1.2). Balachandran and Park [17] have also proved the existence of a mild solution by using the theory of strongly continuous cosine families and the Schaefer fixed point theorem. Bahuguna [6, 7] has established the existence, uniqueness, continuation of a solution to the maximal interval of existence, and the global existence of a strong solution and a classical solution under different conditions.

In this chapter we have shown the convergence of the Faedo-Galerkin approximation to the solution of the abstract second order semi-linear evolution equation (3.1.2). Our aim is to extend for second order semi-linear equation the ideas and techniques of Bahuguna, Srivastava and Singh [16] which are applied to obtain the Faedo-Galerkin approximation of the solution to a first order semi-linear integro-differential equation of the form

$$\begin{aligned} u'(t) + Au(t) &= f(t, u(t)) + \int_0^t a(t-s)g(s, u(s))ds, \quad 0 < t \leq T < \infty, \\ u(0) &= \phi. \end{aligned} \tag{3.1.6}$$

In [16], authors have used the idea of Miletta [72] to establish the convergence of the Faedo-Galerkin approximations of the solution to (3.1.6). Miletta [72] has established results for the first order semi-linear evolution equation

$$u'(t) + Au(t) = M(u(t)), \quad u(0) = \phi,$$

in a separable Hilbert space and A satisfies similar conditions as considered in this Chapter and M is a nonlinear map defined on $D(A^\alpha)$, for some α, $0 < \alpha < 1$, which satisfies a Lipschitz condition on ball in $D(A^\alpha)$.

In [97], Zarubin find estimates for the rate of convergence of the Faedo-Galerkin method for the Cauchy problem $u' + Au + K(t)u = h(t)$, $u(0) = 0$, while in [98], for the problem $u'(t) + A(t)u(t) + K(t, u(t)) = h(t)$, $u(0) = 0$, when $A(t)$ is a strongly continuous and differentiable operator with a domain that does not depend on t, and $K(t, .)$ is a nonlinear Frechet differentiable operator, that is subordinate to A(t) with order less than one.

Despite the widespread use of the Faedo-Galerkin method (in many applications it is referred to as the method of harmonic balance), the convergence behavior in many cases is not known. Bazely [20, 21] showed the uniform convergence of the approximations to solutions of the non-linear wave equation

$$u''(t) + Au(t) + M(u(t)) = 0, \quad u(0) = \phi, \quad u'(0) = \psi,$$

on any closed subinterval $[0, T]$ of existence of the solution.

In [30], Doronin, Lar'kin and Souza have proved the existence of a global generalized solution to initial boundary value problem for the wave equation with nonlinear second order dissipative boundary conditions with the help of the Faedo-Galerkin Method, apriori estimates and compactness arguments. Bessaih and Flandoli [23] investigated a two-dimensional Euler equation subject to a stochastic perturbation (a noise). An existence and uniqueness result is proved under some assumptions of spatial regularity on the noise by using Faedo-Galerkin method. Bellery and Pata [22] have considered the strongly damped semi-linear equation in $\mathbf{R}^3$,

$$u_{tt} - \Delta u_t - \Delta u + g(x, u) + \phi(x)u_t = f(x, t) \quad \text{in} \quad \mathbf{R}^3 \times (0, T],$$

with the initial conditions, $u(x,0) = u_0(x)$, $u_t(x,0) = v_0(x)$, and the condition that $\lim_{|x|\to\infty} |u(x,t)| = 0$ for all $t \in [0,T]$. In this, proof of the existence and uniqueness results is carried out via standard Faedo-Galerkin approximation scheme.

The organization of this chapter is as follows. In the second section we state the preliminaries and the assumptions required to establish the convergence results. We consider an equivalent form of (3.1.2) in a Hilbert space also in this section. In the third section we consider a pair of associated nonlinear integral equations to this equivalent form. The solutions to these integral equations are defined as mild solutions to (3.1.2) or to the equivalent problem. Using the pair of associated nonlinear integral equations and the projection operators, we consider a pair of the approximate nonlinear integral equations and we show the existence and uniqueness of solutions to this pair of approximate integral equations using the Banach contraction principle in this section. In the fourth section we establish the convergence of solutions and convergence of the pair of approximate integral equations to the pair of associated integral equations, limit of the solutions to the approximate integral equations being the solution of the pair of associated integral equations. Further, we show, in this section, that the solution can be extended to the maximal interval of existence and it is unique. Finally, in the fifth section we consider the Faedo-Galerkin approximations of solutions and prove some results concerning the convergence of such approximations.

3.2 Preliminaries and Assumptions

In this section we consider some preliminaries and assumptions essential for our purpose. The operator A and the nonlinear map f assume to satisfy the following assumptions.

(H1) A is a closed, positive definite, self-adjoint, linear operator from the domain of

A, $D(A) \subset H$ into H such that $D(A)$ is dense in H. A has a pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq, \cdots$$

and the corresponding complete orthonormal system of eigen functions $\{u_i\}$, i.e.,

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied, then $-A$ generates an analytic semigroup in H which we denote by e^{-tA}, $t \geq 0$. It follows that the fractional power A^α of A for $0 \leq \alpha \leq 1$ are well defined from $D(A^\alpha) \subseteq H$ into H (cf. Pazy [77], pp. 69-75). $D(A^\alpha)$ for $0 \leq \alpha \leq 1$ is Banach space endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha).$$

Henceforth, we represent by X_α, the space $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$ for $0 \leq \alpha \leq 1$.

(H2) The map f is defined from $[0,\infty) \times X_1 \times X_\alpha$ into H and there exists a nondecreasing function F_R from $[0,\infty)$ into $[0,\infty)$ depending on $R > 0$ such that

$$\begin{aligned} \|f(t,u,v)\| &\leq F_R(t), \\ \|f(t,u_1,v_1) - f(t,u_2,v_2)\| &\leq F_R(t)\{\|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha\}, \end{aligned}$$

for all (t,u,v) and (t,u_1,v_1), (t,u_2,v_2) in $[0,\infty) \times B_R(X_1 \times X_\alpha, (x_0, x_1))$, where

$$B_R(X_1 \times X_\alpha, (x_0,x_1)) = \{(x,y) \in X_1 \times X_\alpha \, : \, \|x - x_0\|_1 + \|y - x_1\|_\alpha \leq R\}.$$

Without loss of generality, we concentrate on the following problem,

$$\begin{aligned} &\frac{d^2u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) = f(t, u(t), \frac{du}{dt}(t)), \quad t > 0, \\ &u(0) = x_0, \quad u'(0) = x_1, \end{aligned} \tag{3.2.1}$$

in a separable Hilbert space H, since the other terms in (3.1.2) may be transferred on the right and merged with f and the changed f still satisfies (H2).

3.3 Approximate integral equations

The existence of solution to (3.2.1) is closely associated with the following pair of integral equations (cf. Bahuguna [6, 7]),

$$\begin{aligned} u(t) &= x_0 + (e^{-tA} - I)(-A)^{-1}x_1 + \int_0^t (e^{-(t-s)A} - I)(-A)^{-1} f(s, u(s), v(s))ds, \quad t \geq 0, \\ v(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A} f(s, u(s), v(s))ds, \quad t \geq 0. \end{aligned} \tag{3.3.1}$$

In this section we will consider a pair of approximate integral equations corresponding to (3.3.1) and establish the existence and uniqueness of the solutions to the pair of approximate integral equations. By a solution (u, v) to (3.3.1) on $[0, T]$, $0 < T < \infty$, we mean a function $(u, v) \in X_1(T) \times X_\alpha(T)$ for some $0 < \alpha < 1$ satisfying (3.3.1), where $X_1(T) \times X_\alpha(T)$ is the Banach space $C([0, T], X_1 \times X_\alpha)$ of all continuous functions from $[0, T]$ into $X_1 \times X_\alpha$ endowed with the supremum norm

$$\|(u, v)\|_{X_1(T) \times X_\alpha(T)} = \|u\|_{X_1(T)} + \|v\|_{X_\alpha(T)},$$

where

$$\|u\|_{X_1(T)} = \sup_{0 \leq t \leq T} \|Au(t)\| = \sup_{0 \leq t \leq T} \|u(t)\|_1$$

and

$$\|v\|_{X_\alpha(T)} = \sup_{0 \leq t \leq T} \|A^\alpha v(t)\| = \sup_{0 \leq t \leq T} \|v(t)\|_\alpha.$$

By a solution (u, v) to (3.3.1) on $[0, \tilde{T})$, $0 < \tilde{T} \leq \infty$, we mean a function $(u, v) \in X_1(T) \times X_\alpha(T)$ for some $0 < \alpha < 1$ satisfying (3.3.1) in $[0, T]$ for every $0 < T < \tilde{T}$.

Since $-A$ generates the analytic semigroup e^{-tA}, $t \geq 0$ we may add cI to $-A$ for a suitable constant c, if necessary, and assume without loss of generality that $\|e^{-tA}\| \leq M$ and $-A$ is invertible. Furthermore, it follows that A^α commutes with e^{-tA} and there exists a constant $C_\alpha > 0$ depending on α such that

$$\|e^{-tA}A^\alpha\| \leq C_\alpha t^{-\alpha}, \quad t > 0.$$

Let $0 < T_0 < \infty$ be arbitrary fixed and

$$L(R) = (1+R)F_{\tilde{R}}(T_0),$$

where

$$\tilde{R} = \tilde{R}_1 + \tilde{R}_2, \quad \tilde{R}_1 = \sqrt{R^2 + \|x_0\|_1^2} \quad \text{and } \tilde{R}_2 = \sqrt{R^2 + \|x_1\|_\alpha^2}.$$

Let $0 < T \leq T_0$ be such that

$$\sup_{0 \leq t \leq T} \{\|(e^{-tA} - I)x_1\| + \|(e^{-tA} - I)A^\alpha x_1\|\} < \frac{R}{3}$$

and

$$T < \min\left\{T_0, \frac{R}{3}(M+1)^{-1}L(R)^{-1}, [\frac{R}{3}C_\alpha^{-1}(1-\alpha)L(R)^{-1}]^{\frac{1}{1-\alpha}}\right\}.$$

Let H_n denote the finite dimensional subspace of H spanned by $\{u_0, u_1, \ldots, u_n\}$ and for $n = 1, 2, \ldots$, let $P^n : H \to H_n$ be the corresponding projection operator. For each n, we define

$$f_n : [0,T] \times X_1(T) \times X_\alpha(T) \to H,$$

such that

$$f_n(t, u, v) = f(t, P^n u(t), P^n v(t)).$$

We set $\tilde{x_0}(t) \equiv x_0$ and $\tilde{x_1}(t) \equiv x_1$ for $t \in [0,T]$. Let $W_R = B_R(X_1(T) \times X_\alpha(T), (\tilde{x_0}, \tilde{x_1}))$, where

$$\begin{aligned} &B_R(X_1(T) \times X_\alpha(T), (\tilde{x_0}, \tilde{x_1})) \\ &\quad = \{(y_1, y_2) \in X_1(T) \times X_\alpha(T) : \|y_1 - \tilde{x_0}\|_{X_1(T)} + \|y_2 - \tilde{x_1}\|_{X_\alpha(T)} \leq R\}. \end{aligned}$$

Define a map S_n on W_R such that $S_n(u,v) := (\hat{u},\hat{v})$ with

$$\begin{aligned}\hat{u}(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}f_n(s,u,v)ds,\\ \hat{v}(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}f_n(s,u,v)ds.\end{aligned} \tag{3.3.2}$$

Propostion 3.3.1 *Let (H1) and (H2) hold. Then there exists unique $(u_n, v_n) \in W_R$ such that $S_n(u_n,v_n) = (u_n,v_n)$ for each $n = 1,2,...$, i.e., (u_n,v_n) satisfies the pair of approximate integral equations*

$$\begin{aligned}u_n(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}f_n(s,u_n,v_n)ds,\\ v_n(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}f_n(s,u_n,v_n)ds.\end{aligned} \tag{3.3.3}$$

Proof. First, we claim that $S_n : W_R \to W_R$. For this we need to show first that the map $t \mapsto (S_n(u,v))(t)$ is continuous from $[0,T]$ into $X_1 \times X_\alpha$ with respect to norm $\|.\|_1 + \|.\|_\alpha$. For $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$\begin{aligned}\|\hat{u}(t_2) - \hat{u}(t_1)\|_1 &+ \|\hat{v}(t_2) - \hat{v}(t_1)\|_\alpha\\ &\le \|(e^{-t_2A} - e^{-t_1A})x_1\| + \|(e^{-t_2A} - e^{-t_1A})x_1\|_\alpha\\ &+ \int_{t_1}^{t_2} \|e^{-(t_2-s)A} - I\|\|f_n(s,u,v)\|ds\\ &+ \int_0^{t_1} \|e^{-(t_2-s)A} - e^{-(t_1-s)A}\|\|f_n(s,u,v)\|ds\\ &+ \int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\|\|f_n(s,u,v)\|ds\\ &+ \int_0^{t_1} \|(e^{-(t_2-s)} - e^{-(t_1-s)})A^\alpha\|\|f_n(s,u,v)\|ds.\end{aligned} \tag{3.3.4}$$

Also, we have

$$\int_{t_1}^{t_2} \|s^{-(t_2-s)A} - I\| \|f_n(s,u,v)\| ds \leq (M+1)F_{\tilde{R}}(T_0)(t_2-t_1), \tag{3.3.5}$$

$$\int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^{\alpha}\| \|f_n(s,u,v)\| ds \leq C_{\alpha}F_{\tilde{R}}(T_0)\frac{(t_2-t_1)^{1-\alpha}}{1-\alpha}. \tag{3.3.6}$$

Part (d) of Theorem 1.3.19 implies that for $0 < \vartheta \leq 1$ and $x \in D(A^{\vartheta})$, we have

$$\|(e^{-tA} - I)x\| \leq C'_{\vartheta}t^{\vartheta}\|x\|_{\vartheta}. \tag{3.3.7}$$

If $0 < \vartheta < 1$ is such that $0 < \alpha + \vartheta < 1$, then $A^{\alpha}y \in D(A^{\vartheta})$ for any $y \in D(A^{\alpha+\vartheta})$. Therefore, for $t, s \in [0, T]$, we have

$$\begin{aligned} \|(e^{-tA} - I)A^{\alpha}e^{-sA}x\| &\leq C'_{\vartheta}t^{\vartheta}\|A^{\alpha}e^{-sA}x\|_{\vartheta} = C'_{\vartheta}t^{\vartheta}\|A^{\alpha+\vartheta}e^{-sA}x\| \\ &\leq C'_{\vartheta}C_{\alpha+\vartheta}t^{\vartheta}s^{-(\alpha+\vartheta)}\|x\|. \end{aligned} \tag{3.3.8}$$

From (3.3.8), we get

$$\begin{aligned} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^{\alpha}\| &= \|(e^{-(t_2-t_1)A} - I)A^{\alpha}e^{-(t_1-s)A}\| \\ &\leq C'_{\vartheta}C_{\alpha+\vartheta}(t_2-t_1)^{\vartheta}(t_1-s)^{-(\alpha+\vartheta)}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{t_1} \|(e^{-(t_2-s)A} &- e^{-(t_1-s)A})A^{\alpha}\| \|f_n(s,u,v)\| ds \\ &\leq C'_{\vartheta}C_{\alpha+\vartheta}F_{\tilde{R}}(T_0)\frac{T_0^{1-(\alpha+\vartheta)}}{1-(\alpha+\vartheta)}(t_2-t_1)^{\vartheta}. \end{aligned} \tag{3.3.9}$$

Also, from (3.3.8), we have

$$\begin{aligned}\|(e^{-tA}-I)e^{-sA}x\| &\leq C'_{\vartheta}t^{\vartheta}\|e^{-sA}x\|_{\vartheta} = C'_{\vartheta}t^{\vartheta}\|A^{\vartheta}e^{-sA}x\|\\ &\leq C'_{\vartheta}C_{\vartheta}t^{\vartheta}s^{-\vartheta}\|x\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|e^{-(t_2-s)A}-e^{-(t_1-s)A}\| &= \|(e^{-(t_2-t_1)A}-I)e^{-(t_1-s)A}\|\\ &\leq C'_{\vartheta}C_{\vartheta}(t_2-t_1)^{\vartheta}(t_1-s)^{-\vartheta}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{t_1}\|e^{-(t_2-s)A} &- e^{-(t_1-s)A}\|\|f_n(s,u,v)\|ds\\ &\leq C'_{\vartheta}C_{\vartheta}F_{\tilde{R}}(T_0)\frac{T_0^{1-\vartheta}}{1-\vartheta}(t_2-t_1)^{\vartheta}. \qquad (3.3.10)\end{aligned}$$

From inequalities (3.3.4), (3.3.5), (3.3.6), (3.3.9) and (3.3.10), it follows that $(S_n(u,v))(t)$ is continuous from $[0,T]$ into $X_1\times X_\alpha$ with respect to norm $\|.\|_1+\|.\|_\alpha$. Now, we show that $S_n(u,v)\in W_R$, i.e., $(\hat{u},\hat{v})\in W_R$. Consider

$$\begin{aligned}\|\hat{u}(t)-x_0\|_1 &+ \|\hat{v}(t)-x_1\|_\alpha\\ &\leq \|(e^{-tA}-I)x_1\|+\|(e^{-tA}-I)A^{\alpha}x_1\|\\ &\quad+\int_0^t\|e^{-(t-s)A}-I\|\|f_n(s,u,v)\|ds+\int_0^t\|e^{-(t-s)A}A^{\alpha}\|\|f_n(s,u,v)\|ds\\ &\leq \frac{R}{3}+(M+1)F_{\tilde{R}}(T_0)T+C_\alpha F_{\tilde{R}}(T_0)\int_0^t(t-s)^{-\alpha}ds\\ &\leq \frac{R}{3}+(M+1)L(R)T+C_\alpha L(R)\frac{T^{1-\alpha}}{1-\alpha}\leq R.\end{aligned}$$

Taking supremum over $[0,T]$, we obtain that S_n maps W_R into W_R. Now, it remains

to show that S_n is contraction on W_R. For $(u_1, v_1), (u_2, v_2) \in W_R$, we have

$$\begin{aligned}\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 &+ \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\ &\leq \int_0^t \|e^{-(t-s)A} - I\| \|f_n(s, u_1, v_1) - f_n(s, u_2, v_2)\| ds \\ &+ \int_0^t \|e^{-(t-s)A} A^\alpha\| \|f_n(s, u_1, v_1) - f_n(s, u_2, v_2)\| ds.\end{aligned}$$

From (H2), we have

$$\begin{aligned}\|f_n(s, u_1, v_1) &- f_n(s, u_2, v_2)\| \\ &= \|f(s, P^n u_1(s), P^n v_1(s)) - f(s, P^n u_2(s), P^n v_2(s))\| \\ &\leq F_{\tilde{R}}(T_0)(\|P^n u_1(s) - P^n u_2(s)\|_1 + \|P^n v_1(s) - P^n v_2(s)\|_\alpha) \\ &\leq F_{\tilde{R}}(T_0)(\|u_1(s) - u_2(s)\|_1 + \|v_1(s) - v_2(s)\|_\alpha) \\ &\leq F_{\tilde{R}}(T_0)(\|u_1 - u_2\|_{X_1(T)} + \|v_1 - v_2\|_{X_\alpha(T)}).\end{aligned}$$

Hence

$$\begin{aligned}\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 &+ \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\ &\leq \left\{(M+1)F_{\tilde{R}}(T_0)T + C_\alpha F_{\tilde{R}}(T_0) \int_0^t (t-s)^{-\alpha} ds\right\} \\ &\quad \times (\|u_1 - u_2\|_{X_1(T)} + \|v_1 - v_2\|_{X_\alpha(T)}) \\ &\leq \frac{1}{R}\left\{(M+1)L(R)T + C_\alpha L(R)\frac{T^{1-\alpha}}{1-\alpha}\right\} \\ &\quad \times (\|u_1 - u_2\|_{X_1(T)} + \|v_1 - v_2\|_{X_\alpha(T)}) \\ &\leq \frac{2}{3}(\|u_1 - u_2\|_{X_1(T)} + \|v_1 - v_2\|_{X_\alpha(T)}).\end{aligned}$$

Taking supremum over $[0, T]$, we get

$$\|\hat{u}_1 - \hat{u}_2\|_{X_1(T)} + \|\hat{v}_1 - \hat{v}_2\|_{X_\alpha(T)} \leq \frac{2}{3}(\|u_1 - u_2\|_{X_1(T)} + \|v_1 - v_2\|_{X_\alpha(T)}).$$

Thus, S_n is a strict contraction on W_R. Hence, there exists a unique $(u_n, v_n) \in W_R$ such that $S_n(u_n, v_n) = (u_n, v_n)$. Clearly, (u_n, v_n) satisfies (3.3.3). This completes the proof of the proposition.

Propostion 3.3.2 *Let (H1) and (H2) hold. If $(x_0, x_1) \in D(A) \times D(A^\alpha)$ for some $0 < \alpha < 1$, then $(u_n(t), v_n(t)) \in D(A) \times D(A^\vartheta)$ for all $t \in (0, T]$ where $0 \le \vartheta < 1$, Furthermore, if $(x_0, x_1) \in D(A) \times D(A)$, then $(u_n(t), v_n(t)) \in D(A) \times D(A^\vartheta)$ for all $t \in [0, T]$ where $0 \le \vartheta < 1$.*

Proof. From the Proposition 3.3.1 we have the existence of a unique $(u_n, v_n) \in W_R$ satisfying (3.3.3). Clearly, $u_n(t) \in D(A)$. Part (a) of Theorem 1.3.19 implies that $e^{-tA} : H \to D(A^\vartheta)$ for $t > 0$ and $0 \le \vartheta < 1$. Also, from Theorem 1.3.6, we have $e^{-tA}x \in D(A)$ if $x \in D(A)$ and for $x \in H$, $\int_0^t e^{-tA}x ds \in D(A)$. The result of the proposition follows from these fact and the fact that $D(A) \subseteq D(A^\vartheta)$ for $0 \le \vartheta \le 1$.

Propostion 3.3.3 *Let (H1) and (H2) hold. Then for any $x_1 \in D(A^\alpha)$, $0 < \alpha < 1$ and any $t_0 \in (0, T]$, there exists a constant V_{t_0}, independent of n, such that*

$$\|v_n(t)\|_\vartheta \;\le\; V_{t_0}, \quad 0 \le \vartheta < 1, \quad t_0 \le t \le T.$$

Moreover, if $x_1 \in D(A)$, then there exists a constant V_0 independent of n, such that

$$\|v_n(t)\|_\vartheta \;\le\; V_0, \quad 0 \le \vartheta < 1, \quad 0 \le t \le T.$$

Proof. Applying A^ϑ on both the sides of second integral equation of (3.3.3) and using part (c) of Theorem (1.3.19), we have for $t_0 \le t \le T$,

$$\begin{aligned}
\|v_n(t)\|_\vartheta &\le \|A^\vartheta e^{-tA} x_1\| + \int_o^t \|e^{-(t-s)A} A^\vartheta\| \|f_n(s, u_n, v_n)\| ds \\
&\le C_\vartheta t^{-\vartheta} \|x_1\| + C_\vartheta F_{\tilde{R}}(T_0) \frac{T^{1-\vartheta}}{1-\vartheta} \\
&\le C_\vartheta t_0^{-\vartheta} \|x_1\| + C_\vartheta L(R) \frac{T^{1-\vartheta}}{1-\vartheta} \;\le\; V_{t_0}.
\end{aligned}$$

Since $x_1 \in D(A)$ implies that $x_1 \in D(A^\vartheta)$ for $0 \le \vartheta < 1$, we get

$$\begin{aligned} \|v_n(t)\|_\vartheta &\le \|e^{-tA}A^\vartheta x_1\| + C_\vartheta L(R)\frac{T^{1-\vartheta}}{1-\vartheta} \\ &\le M\|A^\vartheta x_1\| + C_\vartheta L(R)\frac{T^{1-\vartheta}}{1-\vartheta} \le V_0. \end{aligned}$$

This completes the proof of the proposition.

3.4 Convergence of approximate solutions

In this section we establish the convergence of the solution $(u_n, v_n) \in X_1(T) \times X_\alpha(T)$ of approximate integral equations

$$\begin{aligned} u_n(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1} f_n(s, u_n, v_n)ds, \\ v_n(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A} f_n(s, u_n, v_n)ds, \end{aligned} \tag{3.4.1}$$

to a unique solution (u, v) of (3.3.1). For proving the convergence, we need following stronger assumption than (H2) on the nonlinear map f.

(H2′) The map f is defined from $[0,\infty) \times X_1 \times X_\alpha$ into X_β for $0 < \alpha < \beta < 1$ and there exists a nondecreasing function $\tilde{F}_R$ from $[0,\infty)$ into $[0,\infty)$ depending on $R > 0$ such that

$$\begin{aligned} \|f(t,u,v)\|_\beta &\le \tilde{F}_R(t), \\ \|f(t,u_1,v_1) - f(t,u_2,v_2)\|_\beta &\le \tilde{F}_R(t)\{\|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha\}, \end{aligned}$$

for all (t,u,v) and (t,u_1,v_1), (t,u_2,v_2) in $[0,\infty) \times B_R(X_1 \times X_\alpha, (x_0, x_1))$.

We can easily check that the changed f in (3.2.1) still satisfies (H2) with $F_R(t) = C_1\tilde{F}_R(t) + C_2$ for some constant C_1 and C_2 but this changed f does not satisfy (H2′).

Propostion 3.4.1 *Let (H1) and (H2′) hold. If $(x_0, x_1) \in D(A) \times D(A^\alpha)$, $0 < \alpha < 1$, then for any $t_0 \in (0, T]$*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ t_0\leq t\leq T\}} \{\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha\} = 0.$$

Proof. For $n \geq m$, we have

$$\begin{aligned}
\|f_n(t, u_n, v_n) &- f_m(t, u_m, v_m)\| \\
&\leq \|f_n(t, u_n, v_n) - f_n(t, u_m, v_m)\| + \|f_n(t, u_m, v_m) - f_m(t, u_m, v_m)\| \\
&\leq \|f(t, P^n u_n(t), P^n v_n(t)) - f(t, P^n u_m(t), P^n v_m(t))\| \\
&\quad + \|f(t, P^n u_m(t), P^n v_m(t)) - f(t, P^m u_m(t), P^m v_m(t))\| \\
&\leq F_{\tilde{R}}(T_0)[\|P^n u_n(t) - P^n u_m(t)\|_1 + \|P^n v_n(t) - P^n v_m(t)\|_\alpha \\
&\quad + \|(P^n - P^m)u_m(t)\|_1 + \|(P^n - P^m)v_m(t)\|_\alpha] \\
&\leq F_{\tilde{R}}(T_0)[\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha \\
&\quad + \|(P^n - P^m)u_m(t)\|_1 + \|(P^n - P^m)v_m(t)\|_\alpha].
\end{aligned}$$

Also,

$$\begin{aligned}
\|(P^n - P^m)v_m(t)\|_\alpha &= \|A^\alpha(P^n - P^m)v_m(t)\| \\
&= \|A^{\alpha-\vartheta}(P^n - P^m)A^\vartheta v_m(t)\| \\
&\leq \frac{1}{\lambda_m^{\vartheta-\alpha}}\|(P^n - P^m)A^\vartheta v_m(t)\| \\
&\leq \frac{\|A^\vartheta v_m(t)\|}{\lambda_m^{\vartheta-\alpha}},
\end{aligned}$$

where $\alpha < \vartheta < 1$. Thus, we have

$$\begin{aligned}
\|f_n(t, u_n, v_n) &- f_m(t, u_m, v_m)\| \\
&\leq L(R)[\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha \\
&\quad + \|(P^n - P^m)u_m(t)\|_1 + \frac{\|v_m(t)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}}].
\end{aligned}$$

For convenience, we denote

$$\xi_{m,n}(t) = \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha.$$

For $0 < t_0' < t_0$, we may write

$$\begin{aligned}
\xi_{m,n}(t) \;&\leq\; \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right) \|e^{-(t-s)A} - I\|\|f_n(s, u_n, v_n) - f_m(s, u_m, v_m)\|ds \\
&+ \left(\int_0^{t_0'} + \int_{t_0'}^{t}\right) \|e^{-(t-s)A}A^\alpha\|\|f_n(s, u_n, v_n) - f_m(s, u_m, v_m)\|ds.
\end{aligned}$$

We estimate the first and the third integrals as

$$\int_0^{t_0'} \|e^{-(t-s)A} - I\|\|f_n(s, u_n, v_n) - f_m(s, u_m, v_m)\|ds \;\leq\; 2L(R)(M+1)t_0',$$

$$\int_0^{t_0'} \|e^{-(t-s)A}A^\alpha\|\|f_n(s, u_n, v_n) - f_m(s, u_m, v_m)\|ds \;\leq\; 2L(R)C_\alpha(t_0 - t_0')^{-\alpha}t_0'.$$

We estimate the second integral as

$$\begin{aligned}
&\int_{t_0'}^{t} \|e^{-(t-s)A} - I\|\|f_n(s, u_n, v_n) - f_m(s, u_m, v_m)\|ds \\
&\leq\; (M+1)L(R)\int_{t_0'}^{t}\left(\xi_{m,n}(s) + \|(P^n - P^m)u_m(s)\|_1 + \frac{\|v_m(s)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}}\right) ds \\
&\leq\; (M+1)L(R)\left(\sup_{t_0'\leq s\leq T}\|(P^n - P^m)u_m(s)\|_1 T + \frac{V_{t_0'}}{\lambda_m^{\vartheta-\alpha}}T + \int_{t_0'}^{t}\xi_{m,n}(s)ds\right) \\
&\leq\; (M+1)L(R)\left(B_{m,n}T + \int_{t_0'}^{t}\xi_{m,n}(s)\right),
\end{aligned}$$

where

$$B_{m,n} = \sup_{t_0'\leq s\leq T}\|(P^n - P^m)u_m(s)\|_1 + \frac{V_{t_0'}}{\lambda_m^{\vartheta-\alpha}}.$$

For the fourth integral, we have the estimate

$$\begin{aligned}
&\int_{t_0'}^{t} \|e^{-(t-s)A}A^{\alpha}\|\|f_n(s,u_n,v_n) - f_m(s,u_m,v_m)\|ds \\
&\leq C_{\alpha}L(R)\int_{t_0'}^{t}(t-s)^{-\alpha}\left(\xi_{m,n}(s) + \|(P^n - P^m)u_m(s)\|_1 + \frac{\|v_m(s)\|_{\vartheta}}{\lambda_m^{\vartheta-\alpha}}\right)ds \\
&\leq C_{\alpha}L(R)\left(B_{m,n}\frac{T^{1-\alpha}}{1-\alpha} + \int_{t_0'}^{t}\frac{1}{(t-s)^{\alpha}}\xi_{m,n}(s)\right)ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\xi_{m,n}(t) &\leq 2L(R)[(M+1) + C_{\alpha}(t_0 - t_0')^{-\alpha}]t_0' + C(R,T)B_{m,n} \\
&\quad + L(R)\int_{t_0'}^{t}[(M+1) + \frac{C_{\alpha}}{(t-s)^{\alpha}}]\xi_{m,n}(s)ds \\
&\leq 2L(R)[(M+1) + C_{\alpha}(t_0 - t_0')^{-\alpha}]t_0' + C(R,T)B_{m,n} \\
&\quad + N_1(T^{\alpha}+1)\int_{t_0'}^{t}\frac{1}{(t-s)^{\alpha}}\xi_{m,n}(s)ds,
\end{aligned}$$

where

$$C(R,T) = L(R)\left[(M+1)T + C_{\alpha}\frac{T^{1-\alpha}}{1-\alpha}\right]$$

and

$$N_1 = \max\{(M+1), C_{\alpha}\}L(R).$$

Hence, from Lemma 1.3.27, there exists a constant K such that

$$\xi_{m,n}(t) \leq \{2L(R)[(M+1) + C_{\alpha}(t_0 - t_0')^{-\alpha}]t_0' + C(R,T)B_{m,n}\}K.$$

$B_{m,n} \to 0$ as $m \to \infty$ provided $\|(P^n - P^m)u_m(t)\|_1 \to 0$ as $m \to \infty$ for $0 \leq t \leq T$, we get

$$\lim_{m\to\infty}\sup_{\{n\geq m,\ t_0\leq t\ \leq\ T\}}\xi_{m,n}(t) \leq 2L(R)[(M+1) + C_{\alpha}(t_0 - t_0')^{-\alpha}]Kt_0'.$$

As t_0' is arbitrary, we observe that the right hand side may be made as small as desired by taking t_0' sufficiently small.

Now, we show that for $0 \leq t \leq T$, $\|(P^n - P^m)u_m(t)\|_1 \to 0$ as $m \to \infty$. We can easily check that for $x \in H$ and $\eta < 0$,

$$\|A^\eta(P^n - P^m)x\| \leq \lambda_m^\eta \|(P^n - P^m)x\| \leq \lambda_m^\eta \|x\|. \tag{3.4.2}$$

From equation (3.4.1), we get

$$\begin{aligned} \|A(P^n - P^m)u_m(t)\| &\leq \|(P^n - P^m)Ax_0\| + (M+1)\|(P^n - P^m)x_1\| \\ &\quad + (M+1)\int_0^t \|(P^n - P^m)f_m(s, u_m, v_m)\|ds. \end{aligned} \tag{3.4.3}$$

We note that in (3.2.1), $f(t,u,v) = f^O(t,u,v) - Bu$ where f^O represents original f appears in (3.1.2). Also, we can check $\|Bu\| \leq C\|Au\|$ for $u \in D(A)$ and some constant C. Therefore, we have

$$\begin{aligned} \|(P^n - P^m)f_m(s, u_m, v_m)\| &\leq \|(P^n - P^m)f_m^O(s, u_m, v_m)\| \\ &\quad + \|A(P^n - P^m)u_m(s)\|. \end{aligned} \tag{3.4.4}$$

Since $A^\beta f_m^O(s, u_m, v_m) \in H$, hence from (3.4.2), we have

$$\begin{aligned} \|(P^n - P^m)f_m^O(s, u_m, v_m)\| &= \|A^{-\beta}(P^n - P^m)A^\beta f_m^O(s, u_m, v_m)\| \\ &\leq \frac{1}{\lambda_m^\beta}\|A^\beta f_m^O(s, u_m, v_m)\| \\ &\leq \frac{1}{\lambda_m^\beta}\tilde{F}_{\tilde{R}}(T_0). \end{aligned} \tag{3.4.5}$$

Using (5.4.10) and (3.4.5) in (3.4.3), we get

$$\begin{aligned} \|(P^n - P^m)u_m(t)\|_1 &\leq \|(P^n - P^m)Ax_0\| + (M+1)[\|(P^n - P^m)x_1\| \\ &\quad + \frac{1}{\lambda_m^\beta}T\tilde{F}_{\tilde{R}}(T_0) + \int_0^t \|(P^n - P^m)u_m(s)\|_1 ds]. \end{aligned}$$

Gronwall's inequality implies that

$$\begin{aligned}\|(P^n - P^m)u_m(t)\|_1 \;&\leq\; (\|(P^n - P^m)Ax_0\| + (M+1)[\|(P^n - P^m)x_1\| \\ &+ \frac{1}{\lambda_m^\beta} T\tilde{F}_{\tilde{R}}(T_0)])e^{(M+1)T}\end{aligned}$$

which tend to zero as $m \to \infty$ for $0 \leq t \leq T$. This completes the proof of the proposition.

Corollary 3.4.2 *Let (H1) and (H2′) hold. If $(x_0, x_1) \in D(A) \times D(A)$, then*

$$\lim_{m\to\infty} \sup_{\{n \geq m,\ 0 \leq t \leq T\}} \{\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha\} = 0.$$

Proof. Proposition 3.3.2 and 3.3.3 imply that in the proof of Proposition 3.4.1, we may take $t_0 = 0$.

For the convergence of the solution $(u_n(t), v_n(t))$ of the pair of approximate equation (3.4.1), we have the following result.

Theorem 3.4.3 *Let (H1) and (H2′) hold and let $(x_0, x_1) \in D(A) \times D(A^\alpha)$. Then there exists function $(u, v) \in X_1(T) \times X_\alpha(T)$ such that $(u_n, v_n) \to (u, v)$ as $n \to \infty$ in $X_1(T) \times X_\alpha(T)$ and (u, v) satisfies (3.3.1) on $[0, T]$. Furthermore, (u, v) can be extended to the maximal interval of existence $[0, t_{max})$, $0 < t_{max} \leq \infty$ satisfying (3.3.1) and (u, v) is a unique solution to (3.3.1) on $[0, t_{max})$.*

Proof. First, we assume $(x_0, x_1) \in D(A) \times D(A)$. Corollary 3.4.2 implies that there exists $(u, v) \in X_1(T) \times X_\alpha(T)$ such that (u_n, v_n) converges to (u, v) in $X_1(T) \times X_\alpha(T)$.

Since $(u_n, v_n) \in W_R$, for each n, (u, v) is also in W_R. Further, we have

$$\begin{aligned}
\|f_n(t, u_n, v_n) &- f(t, u(t), v(t))\| \\
&= \|f(t, P^n u_n(t), P^n v_n(t)) - f(t, u(t), v(t))\| \\
&\leq \|f(t, P^n u_n(t), P^n v_n(t)) - f(t, P^n u(t), P^n v(t))\| \\
&\quad + \|f(t, P^n u(t), P^n v(t)) - f(t, u(t), v(t))\| \\
&\leq F_{\tilde{R}}(T_0)[\|P^n u_n(t) - P^n u(t)\|_1 + \|P^n v_n(t) - P^n v(t)\|_\alpha \\
&\quad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha] \\
&\leq F_{\tilde{R}}(T_0)[\|u_n(t) - u(t)\|_1 + \|v_n(t) - v(t)\|_\alpha \\
&\quad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha].
\end{aligned}$$

Taking supremum over $[0, T]$, we get

$$\begin{aligned}
\sup_{0 \leq t \leq T} \|f_n(t, u_n, v_n) &- f(t, u(t), v(t))\| \\
&\leq F_{\tilde{R}}(T_0)[\|u_n - u\|_{X_1(T)} + \|v_n - v\|_{X_\alpha(T)} \\
&\quad + \|(P^n - I)u\|_{X_1(T)} + \|(P^n - I)v\|_{X_\alpha(T)}],
\end{aligned} \tag{3.4.6}$$

and the right hand side of (3.4.6) tends to zero as n tend to infinity. Using (3.4.6) and the bounded convergence theorem in (3.4.1) , we get

$$\begin{aligned}
u(t) &= x_0 - (e^{-tA} - I)(A)^{-1} x_1 - \int_0^t (e^{-(t-s)A} - I)(A)^{-1} f(s, u(s), v(s)) ds, \\
v(t) &= e^{-tA} x_1 + \int_0^t e^{-(t-s)A} f(s, u(s), v(s)) ds.
\end{aligned}$$

Thus, for $(x_0, x_1) \in D(A) \times D(A)$, there exists a unique pair of functions $(u, v) \in X_1(T) \times X_\alpha(T)$ such that $(u_n, v_n) \to (u, v)$ as $n \to \infty$ in $X_1(T) \times X_\alpha(T)$ and (u, v) satisfies (3.3.1) on $[0, T]$.

Now, let us assume that $(x_0, x_1) \in D(A) \times D(A^\alpha)$. Since for $0 < t \leq T$, $(Au_n(t), A^\alpha v_n(t)) \to (Au(t), A^\alpha v(t))$ and also $(u_n(0), v_n(0)) = (u(0), v(0)) = (x_0, x_1)$ we have, for $0 \leq t \leq T$, $(Au_n(t), A^\alpha v_n(t)) \to (Au(t), A^\alpha v(t))$ in H. Furthermore, since each (u_n, v_n) is in W_R, we have $(u, v) \in W_R$ and for any $0 < t_0 \leq T$,

$$\lim_{n\to\infty} \sup_{t_0 \leq t \leq T} \{\|u_n(t) - u(t)\|_1 + \|v_n(t) - v(t)\|_\alpha\} = 0.$$

To prove (u, v) satisfy integral equation (3.3.1), it is sufficient to show that v satisfies second integral equation of (3.3.1). We have

$$\begin{aligned}\sup_{t_0 \leq t \leq T} \|f_n(t, u_n, v_n) - f(t, u(t), v(t))\| &\leq F_{\tilde{R}}(T_0)[\|u_n - u\|_{X_1(T)} + \|v_n - v\|_{X_\alpha(T)} \\ &\quad + \|(P^n - I)u\|_{X_1(T)} + \|(P^n - I)v\|_{X_\alpha(T)}]\end{aligned}$$

and the right hand side of the above inequality tends to zero as n tends to infinity. Now, for $0 < t_0 < t$, we may rewrite the second integral equation of (3.4.1) as

$$v_n(t) = e^{-tA}x_1 + \left(\int_0^{t_0} + \int_{t_0}^{t}\right) e^{-(t-s)A} f_n(s, u_n, v_n)ds.$$

The first integral can be estimated as

$$\left\|\int_0^{t_0} e^{-(t-s)A} f_n(s, u_n, v_n)ds\right\| \leq ML(R)t_0.$$

Therefore,

$$\begin{aligned}\left\|v_n(t) - e^{-tA}x_1 - \int_{t_0}^{t} e^{-(t-s)A} f_n(s, u_n, v_n)ds\right\| &= \left\|\int_0^{t_0} e^{-(t-s)A} f_n(s, u_n, v_n)ds\right\| \\ &\leq ML(R)t_0.\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\left\|v(t) - e^{-tA}x_1 - \int_{t_0}^{t} e^{-(t-s)A} f(s, u(s), v(s))ds\right\| \leq ML(R)t_0.$$

Since $0 < t_0 \leq T$ is arbitrary, we get that v satisfies the second integral equation of (3.3.1).

If (u,v) satisfy (3.3.1) on $[0,T]$, then we show that (u,v) can be extended further. Since $0<T_0<\infty$ was arbitrary, we assume that $0<T<T_0$. We consider the equation

$$\begin{aligned}&U''(t)+AU'(t)=F(t,U(t),U'(t)),\quad t>0,\\&U(0)=u(T),\quad U'(0)=u'(T),\end{aligned}$$

which can also be written as system of equations

$$\begin{aligned}U'(t)&=V(t),\quad U(0)=u(T),\\V'(t)+AV(t)&=F(t,U(t),V(t)),\quad V(0)=v(T).\end{aligned}$$

where $F:[0,T_0-T]\times X_1\times X_\alpha\to H$ are defined by

$$F(t,x,\tilde{x})=f(t+T,x,\tilde{x}),$$

for $(t,x,\tilde{x})\in[0,T_0-T]\times X_1\times X_\alpha$. F satisfies (H2) for T_0 replaced by T_0-T. Hence, there exists a $(U,V)\in C([0,T_1],X_1\times X_\alpha)$, for some $0<T_1\le T_0-T$, satisfying the integral equation

$$\begin{aligned}U(t)&=u(T)+(e^{-tA}-I)(-A)^{-1}v(T)\\&\quad+\int_0^t(e^{-(t-s)A}-I)(-A)^{-1}F(s,U(s),V(s))ds,\\V(t)&=e^{-tA}v(T)+\int_0^t e^{-(t-s)A}F(s,U(s),V(s))ds,\end{aligned}$$

with $0\le t\le T_1$. Now, we define

$$(\tilde{u}(t),\tilde{v}(t))=\begin{cases}(u(t),v(t)), & 0\le t\le T,\\(U(t-T),V(t-T)), & T\le t\le T_1+T.\end{cases}$$

Then, for $t \in [0, T_1 + T]$, $(\tilde{u}, \tilde{v})$ satisfies the integral equations

$$\begin{aligned} \tilde{u}(t) &= x_0 + (e^{-tA} - I)(-A)^{-1}x_1 \\ &\quad + \int_0^t (e^{-(t-s)A} - I)(-A)^{-1} f(s, \tilde{u}(s), \tilde{v}(s))ds, \\ \tilde{v}(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A} f(s, \tilde{u}(s), \tilde{v}(s))ds. \end{aligned} \tag{3.4.7}$$

To see this, we need to verify (3.4.7) only on $[T, T_1 + T]$. We have

$$\begin{aligned} \tilde{u}(t) &= U(t-T) \\ &= u(T) + (e^{-(t-T)A} - I)(-A)^{-1}v(T) \\ &\quad + \int_0^{t-T} (e^{-(t-T-s)A} - I)(-A)^{-1} F(s, U(s), V(s))ds \\ &= x_0 - (e^{-TA} - I)(A)^{-1}x_1 - \int_0^T (e^{-(T-s)A} - I)(A)^{-1} f(s, u(s), v(s))ds \\ &\quad - (e^{-(t-T)A} - I)(A)^{-1} \left(e^{-TA}x_1 + \int_0^T e^{-(T-s)A} f(s, u(s), v(s))ds \right) \\ &\quad - \int_0^{t-T} (e^{-(t-T-s)A} - I)(A)^{-1} F(s, U(s), V(s))ds. \end{aligned}$$

Putting $T + s = \eta$, we have

$$\begin{aligned} \tilde{u}(t) &= x_0 - (e^{-tA} - I)(A)^{-1}x_1 - \int_0^T (e^{-(t-s)A} - I)(A)^{-1} f(s, u(s), v(s))ds \\ &\quad - \int_T^t e^{-(t-s)A} - I)(A)^{-1} F(s-T, U(s-T), V(s-T))ds. \end{aligned}$$

Since $F(s-T, U(s-T), V(s-T)) = f(s, U(s-T), V(s-T))$, we have

$$\tilde{u}(t) = x_0 - (e^{-tA} - I)(A)^{-1}x_1 - \int_0^T (e^{-(t-s)A} - I)(A)^{-1} f(s, \tilde{u}(s), \tilde{v}(s))ds.$$

Since

$$\begin{aligned}&\|f(s,u_1(s),v_1(s))-f(s,u_2(s),v_2(s))\|\\&\qquad\leq\; F_R(t_{max})[\|u_1(s)-u_2(s)\|_1+\|v_1(s)-v_2(s)\|_\alpha],\end{aligned}$$

we get

$$\begin{aligned}\|u_1(t)-u_2(t)\|_1 \;&+\; \|v_1(t)-v_2(t)\|_\alpha\\&\leq\; (M+1)F_R(t_{max})\int_0^t[\|u_1-u_2\|_{X_1(s)}+\|v_1-v_2\|_{X_\alpha(s)}]ds\\&\quad+F_R(t_{max})C_\alpha\int_0^t(t-s)^{-\alpha}[\|u_1-u_2\|_{X_1(s)}+\|v_1-v_2\|_{X_\alpha(s)}]ds\\&\leq\; N_2(T^\alpha+1)\int_0^t\frac{1}{(t-s)^\alpha}[\|u_1-u_2\|_{X_1(s)}+\|v_1-v_2\|_{X_\alpha(s)}]ds,\end{aligned}$$

where

$$N_2=\max\{(M+1),C_\alpha\}F_R(t_{max}).$$

Hence, from Lemma 1.3.27, we get

$$\|u_1(t)-u_2(t)\|_1+\|v_1(t)-v_2(t)\|_\alpha=0,$$

for all $0\leq t\leq T$. From the facts that

$$\|u_1(t)-u_2(t)\| \;\leq\; \frac{1}{\lambda_0}\|u_1(t)-u_2(t)\|_1$$

and

$$\|v_1(t)-v_2(t)\| \;\leq\; \frac{1}{\lambda_0^\alpha}\|v_1(t)-v_2(t)\|_\alpha,$$

it follows that $(u_1,v_1)=(u_2,v_2)$ on $[0,T]$. Since $0<T<t_{max}$ was arbitrary, we have $(u_1,v_1)=(u_2,v_2)$ on $[0,t_{max})$. This completes the proof of the theorem.

3.5 Faedo-Galerkin approximations

For any $0 < T < t_{max}$, we have a unique pair $(u, v) \in X_1(T) \times X_\alpha(T)$ satisfying the integral equations

$$\begin{aligned} u(t) &= x_0 - (e^{-tA} - I)(A)^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)(A)^{-1} f(s, u(s), v(s))ds, \\ v(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A} f(s, u(s), v(s))ds. \end{aligned} \tag{3.5.1}$$

Also, we have a unique solution $(u_n, v_n) \in X_1(T) \times X_\alpha(T)$ of the approximate integral equations

$$\begin{aligned} u_n(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1} f(s, P^n u_n(s), P^n v_n(s))ds, \\ v_n(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A} f(s, P^n u_n(s), P^n v_n(s))ds. \end{aligned} \tag{3.5.2}$$

If we project the equation (3.5.2) onto H_n, we get the Faedo-Galerkin approximations $(\hat{u}_n(t), \hat{v}_n(t)) = (P^n u_n(t), P^n v_n(t))$ satisfying

$$\begin{aligned} \hat{u}_n(t) &= P^n x_0 - (e^{-tA} - I)A^{-1}P^n x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}P^n f(s, \hat{u}_n(s), \hat{v}_n(s))ds, \\ \hat{v}_n(t) &= e^{-tA}P^n x_1 + \int_0^t e^{-(t-s)A} P^n f(s, \hat{u}_n(s), \hat{v}_n(s))ds. \end{aligned} \tag{3.5.3}$$

The solution (u, v) of (3.5.1) and $(\hat{u}_n, \hat{v}_n)$ of (3.5.3), have the representation

$$\begin{aligned} u(t) &= \sum_{i=0}^{\infty} \alpha_i(t) u_i, \quad \alpha_i(t) = (u(t), u_i), \quad i = 0, 1, \ldots; \\ v(t) &= \sum_{i=0}^{\infty} \beta_i(t) u_i, \quad \beta_i(t) = (v(t), u_i), \quad i = 0, 1, \ldots; \end{aligned} \tag{3.5.4}$$

and

$$\hat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) u_i, \quad \alpha_i^n(t) = (\hat{u}_n(t), u_i), \quad i = 0, 1, \ldots, n;$$
$$\hat{v}_n(t) = \sum_{i=0}^{n} \beta_i^n(t) u_i, \quad \beta_i^n(t) = (\hat{v}_n(t), u_i), \quad i = 0, 1, \ldots, n. \tag{3.5.5}$$

Using (3.5.5) in (3.5.3), we obtain the following system of first order integro-differential and differential equations,

$$\begin{aligned} \frac{d\alpha_i^n(t)}{dt} + \lambda_i \alpha_i^n(t) &= \lambda_i \phi_i + \psi_i + \int_0^t F_i^n(s, \alpha_0^n(s), \ldots, \alpha_n^n(s), \beta_0^n(s), \ldots, \beta_n^n(s)) ds, \\ \frac{d\beta_i^n(t)}{dt} + \lambda_i \beta_i^n(t) &= F_i^n(t, \alpha_0^n(t), \ldots, \alpha_n^n(t), \beta_0^n(t), \ldots, \beta_n^n(t)), \end{aligned} \tag{3.5.6}$$

with the initial conditions

$$\alpha_i^n(0) = \phi_i, \quad \beta_i^n(0) = \psi_i,$$

where

$$F_i^n(t, \alpha_0^n(t), \ldots, \alpha_n^n(t), \beta_0^n(t), \ldots, \beta_n^n(t)) = \left(f(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i, \sum_{i=0}^{n} \beta_i^n(t) u_i), u_i \right),$$

and $\phi_i = (x_0, u_i)$, $\psi_i = (x_1, u_i)$ for $i = 1, 2, \ldots\ldots n$.

The systems (3.5.6) determine the $\alpha_i^n(t)$'s and $\beta_i^n(t)$'s. Now, we shall show the convergence of (α_i^n, β_i^n) to (α, β). It can be easily checked that

$$\begin{aligned} A[u(t) - \hat{u}(t)] &= A\left[\sum_{i=0}^{\infty} (\alpha_i(t) - \alpha_i^n(t)) u_i\right] \\ &= \sum_{i=0}^{\infty} \lambda_i (\alpha_i(t) - \alpha_i^n(t)) u_i \end{aligned}$$

and

$$\begin{aligned} A^\alpha[v(t)-\hat{v}(t)] &= A^\alpha\left[\sum_{i=0}^{\infty}(\beta_i(t)-\beta_i^n(t))u_i\right] \\ &= \sum_{i=0}^{\infty}\lambda_i^\alpha(\beta_i(t)-\beta_i^n(t))u_i. \end{aligned}$$

Thus, we have

$$\|A[u(t)-\hat{u}(t)]\|^2 \ \geq \ \sum_{i=0}^{n}\lambda_i^2(\alpha_i(t)-\alpha_i^n(t))^2$$

and

$$\|A^\alpha[v(t)-\hat{v}(t)]\|^2 \ \geq \ \sum_{i=0}^{n}\lambda_i^{2\alpha}(\beta_i(t)-\beta_i^n(t))^2.$$

We have the following convergence Theorem.

Theorem 3.5.1 *Let (H1) and (H2) hold. Then, we have the following.*

(a) If $(x_0,x_1)\in D(A)\times D(A^\alpha)$, then for any $0<t_0\leq T$,

$$\lim_{n\to\infty}\sup_{t_0\leq t\leq T}\left\{\sum_{i=0}^{n}\lambda_i^2(\alpha_i(t)-\alpha_i^n(t))^2+\sum_{i=0}^{n}\lambda_i^{2\alpha}(\beta_i(t)-\beta_i^n(t))^2\right\}=0.$$

(b) If $(x_0,x_1)\in D(A)\times D(A)$, then

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\left\{\sum_{i=0}^{n}\lambda_i^2(\alpha_i(t)-\alpha_i^n(t))^2+\sum_{i=0}^{n}\lambda_i^{2\alpha}(\beta_i(t)-\beta_i^n(t))^2\right\}=0.$$

The assertion of Theorem 3.5.1 follows from the facts mentioned above and from the following proposition.

Propostion 3.5.2 *Let (H1) and (H2) hold and let T be any number such that $0<T<t_{max}$, then we have the following.*

(a) If $(x_0,x_1)\in D(A)\times D(A^\alpha)$, then for any $0<t_0\leq T$,

$$\lim_{m\to\infty}\sup_{\{n\geq m,\ t_0\leq t\leq T\}}\{\|A(\hat{u}_n(t)-\hat{u}_m(t))\|+\|A^\alpha(\hat{v}_n(t)-\hat{v}_m(t))\|\}=0.$$

(b) If $(x_0, x_1) \in D(A) \times D(A)$*, then*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ 0\leq t\leq T\}} \{\|A(\hat{u}_n(t) - \hat{u}_m(t))\| + \|A^\alpha(\hat{v}_n(t) - \hat{v}_m(t))\|\} = 0.$$

Proof. For $n \geq m$, we have

$$\begin{aligned}
&\|A(\hat{u}_n(t) - \hat{u}_m(t))\| + \|A^\alpha(\hat{v}_n(t) - \hat{v}_m(t))\| \\
&= \|A(P^n u_n(t) - P^m u_m(t))\| + \|A^\alpha(P^n v_n(t) - P^m v_m(t))\| \\
&\leq \|AP^n(u_n(t) - u_m(t))\| + \|A(P^n - P^m)u_m(t)\| \\
&\quad + \|A^\alpha P^n(v_n(t) - v_m(t))\| + \|A^\alpha(P^n - P^m)v_m(t)\| \\
&\leq \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha \\
&\quad + \|(P^n - P^m)u_m(t)\|_1 + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\beta v_m\|
\end{aligned}$$

If $(x_0, x_1) \in D(A) \times D(A^\alpha)$, then the result in (a) follows directly from Proposition 3.4.1. If $(x_0, x_1) \in D(A) \times D(A)$, then (b) follows from Corollary 3.4.2.

Chapter 4

Second Order semi-linear integro-differential Equations - I

Obviousness is always the enemy of correctness.
Hence we must invent a new and
difficult symbolism in which nothing
is obvious

— Novalis

4.1 Introduction

In chapters 2 and 3, we have established the results for the approximation of the solution of a Sobolev type evolution equation and a second order semi-linear evolution equation, respectively. For both kind of evolution equations, the existence and uniqueness results had been established by Hernández [39] and Bahuguna [6], respectively. Now, we are interested to establish similar results for a class of second order semi-linear integro-differential equation. For such kind of evolution equation the existence and uniqueness results have not been proved. In this chapter, therefore, we shall establish the existence

and uniqueness of a solution for an abstract second order semi-linear integro-differential equation in a Banach space. Such type of equations arise in the study of viscoelastic material with memory.

Consider the following initial boundary value problem for the strongly damped integro-differential equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) + (aL+bI)\left(\frac{\partial u}{\partial t}\right)(x,t) \; + \; (cL+dI)u(x,t) \; &= \; h(x,t,u(x,t),\frac{\partial u}{\partial t}(x,t)) \\
+ \int_{t_0}^{t} k(t-s)h(x,s,u(x,s),\frac{\partial u}{\partial s}(x,s))ds, \quad &(x,t) \in \Omega \times (0,T), \\
u(x,t_0) = x_0(x), \quad \frac{\partial u}{\partial t}(x,t_0) = x_1(x), \quad x \in \Omega, & \qquad (4.1.1)
\end{aligned}$$

where Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and $Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ be a symmetric second order strongly elliptic differential operator in Ω. Here $a > 0$, b, c, d are constants and h and g are smooth nonlinear functions and k is a locally p-integrable function for $1 < p < \infty$.

We can treat (4.1.1), as a special case of the following initial value problem for abstract second order semi-linear integro-differential equation in a Banach space X

$$\begin{aligned}
&\frac{d^2 u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) + Bu(t) \\
&\quad = f(t,u(t),\frac{du}{dt}(t)) + \int_{t_0}^{t} k(t-s)g(s,u(s),\frac{du}{ds}(s))ds, \quad t > t_0, \\
&u(t_0) = x_0, \quad u'(t_0) = x_1. \qquad (4.1.2)
\end{aligned}$$

For (4.1.1) we take $A = aL$, $B = (cL + dI)$ and $f(t,u,v) = h(t,u,v) - bv$. We assume that $-A$ generates an analytic semigroup $T(t)$ in X. The nonlinear map f and g satisfy Assumption (F) and Assumption (G), respectively, and the kernel k satisfies (K) stated in the next section.

In this chapter, we concentrate on the study of the equation

$$u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t-s)g(s, u(s), u'(s))ds, \quad t > t_0,$$
$$u(t_0) = x_0, \quad u'(t_0) = x_1, \tag{4.1.3}$$

as we can merge Bu with f so that f still satisfied (F).

Duvaut and Lions [31], Glowinski, Lions and Tremolieres [34] have studied particular case of (4.1.1) in which $L = -\triangle$ and $k \equiv 0$, in the context of the theorem of viscoelastic materials. Sandefur [84] studied the second order semi-linear differential equation

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t))$$
$$u(0) = \phi, \quad u'(0) = \psi, \tag{4.1.4}$$

in a Banach space X under the assumptions that the linear operators A and B can be decomposed as $-A = A_1 + A_2$ and $B = A_2A_1$, where A_k generates a c_0-semigroup $T_k(t)$, $k = 1, 2$; and f satisfies a locally Lipschitz condition. He established the local existence and uniqueness of a mild solution to (4.1.4), i.e., there exists a continuous function u on $[0, c]$ for some $c > 0$ such that u satisfies the integral equation

$$\begin{aligned} u(t) \; = \; T_1(t)\phi \; &+ \; \int_0^t T_1(t-\tau)T_2(\tau)(\psi - A_1\phi)d\tau \\ &+ \int_0^t \int_0^\tau T_1(t-\tau)T_2(t-s)f(s, u(s))dsd\tau, \end{aligned}$$

where $\phi \in D(A_1)$. Aviles and Sandefur [3] studied the well-posedness of (4.1.4) under similar conditions.

Engler, Neubrander and Sandefur [32] proved the local existence and uniqueness

of a mild solution to special case of (4.1.3) in which $k \equiv 0$, that is,

$$
\begin{aligned}
& u''(t) + Au'(t) = f(t, u(t), u'(t)), \\
& u(t_0) = x_0, \quad u'(t) = x_1, \qquad (4.1.5)
\end{aligned}
$$

under the assumptions that $-A$ generates an analytic semigroup $T(t)$ in X and f satisfies a condition similar to Assumption (F), where a mild solution on $[t_0, t_1)$, for some $t_1 > t_0$, to (4.1.5) is the first component of a solution $(u(t), v(t))$ of the integral equations

$$
\begin{aligned}
u(t) &= x_0 + (T(t-t_0) - I)(-A)^{-1}x_1 \\
&\quad + \int_{t_0}^{t} (T(t-s) - I)(-A)^{-1} f(s, u(s), v(s))ds, \quad t_0 \le t \le t_1, \\
v(t) &= T(t-t_0)x_1 + \int_{t_0}^{t} T(t-s) f(s, u(s), v(s))ds, \quad t_0 \le t \le t_1.
\end{aligned}
$$

Bahuguna [6] improved the results of [32], by showing that (4.1.5) has a unique classical solution locally, that is, there exists a unique $u \in C^1([t_0, t_1) : X) \cap C^2((t_0, t_1) : X)$ and satisfies (4.1.5) on $[t_0, t_1)$ for some $t_1 > t_0$. Further, he has discussed the continuation of this solution, maximal interval of existence and the global existence.

In this chapter, we show that (4.1.3) has a unique classical solution locally, i.e., there exists a unique $u \in C^1([t_0, t_1) : X) \cap C^2((t_0, t_1) : X)$ and satisfies (4.1.3) on $[t_0, t_1)$ for some $t_1 > t_0$. Further, we discuss the continuation of this solution, maximal interval of existence and the global existence. We achieve these objectives by extending the ideas and techniques used in the proof of Theorems 6.3.1 and 6.3.3 in Pazy [77] concerning the semi-linear equations of the first order to (4.1.3). For the global existence, we require the modified version of Lemma 4.4.2, stated and proved in fourth Section which is originally stated in Pazy [77] as Lemma 5.6.7.

4.2 Preliminaries and Assumptions

Let X be a Banach space and let $-A$ generate the analytic semigroup $T(t)$ in X. we note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A+\alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. This allow us to reduce the general case, in which $-A$ is the infinitesimal generator of an analytic semigroup, to the case where the semigroup is bounded and the generator is invertible. Hence, for convenience, without loss of generality, we assume that $T(t)$ is bounded, that is $\|T(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, i.e., $-A$ is invertible. Here $\rho(-A)$ is the resolvent set of $-A$. It follows that, for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with its domain $D(A^\alpha)$ being dense in X. We denote by X_α the Banach space $D(A^\alpha)$ equipped with the norm

$$\|x\|_\alpha = \|A^\alpha x\|$$

which is equivalent to the graph norm of A^α. For $0 < \alpha < \beta$, we have $X_\beta \subset X_\alpha$ and the embedding is continuous.

We consider the problem

$$\begin{aligned} &u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t-s)g(t, u(t), u'(t))ds, \quad t > t_0, \\ &u(t_0) = x_0, \quad u'(t_0) = x_1. \end{aligned} \tag{4.2.1}$$

On the kernel k we assume the following condition:

(K) Kernel $k \in L^p_{loc}(0, \infty)$ for some $1 < p < \infty$ is locally Hölder continuous on $(0, \infty)$, i.e.,

$$|k(t) - k(s)| \;\leq\; L_k|t-s|^\mu \quad \text{for} \quad s, t \in (0, \infty) \quad \text{and} \quad 0 < \mu < 1.$$

The nonlinear functions f and g are assumed to satisfy the following assumptions:

Let U be an open set in $R_+ \times X_1 \times X_\alpha$.

Assumption (F): A function f said to satisfy the Assumption (F), if for every $(t, x, \tilde{x}) \in U$, there exists a neighborhood $V \subset U$ and constant $L_f \geq 0$, $0 < \vartheta \leq 1$, such that

$$\|f(t_1, x_1, \tilde{x}_1) - f(t_2, x_2, \tilde{x}_2)\| \leq L_f[|t_1 - t_2|^\vartheta + \|x_1 - x_2\|_1 + \|\tilde{x}_1 - \tilde{x}_2\|_\alpha], \quad (4.2.2)$$

for all $(t_i, x_i \tilde{x}_i) \in V$.

Assumption (G): A function g said to satisfy the Assumption (G), if for every $(t, x, \tilde{x}) \in U$, there exists a neighborhood $V \subset U$ and a nonnegative function $L_g \in L^q_{loc}(0, \infty)$, where $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\|g(t, x_1, \tilde{x}_1) - g(t, x_2, \tilde{x}_2)\| \leq L_g(t)[\|x_1 - x_2\|_1 + \|\tilde{x}_1 - \tilde{x}_2\|_\alpha], \quad (4.2.3)$$

for all $(t_i, x_i \tilde{x}_i) \in V$.

By a *local classical solution* to (4.2.1), we mean a function $u \in C^1([t_0, t_1) : X) \cap C^2((t_0, t_1) : X)$, satisfying (4.2.1) on $[t_0, t_1)$, for some $t_1 > t_0$. By a *local mild solution* to (4.2.1) on $[t_0, t_1)$, for some $t_1 > t_0$, we mean the first component of a solution (u, v) to integral equations

$$\begin{aligned} u(t) &= x_0 + (T(t - t_0) - I)(-A)^{-1}x_1 + \int_{t_0}^t (T(t - s) - I)(-A)^{-1}[f(s, u(s), v(s)) \\ &\quad + \int_{t_0}^s k(s - \tau)g(\tau, u(\tau), v(\tau))d\tau]ds, \quad t_0 \leq t \leq t_1, \\ v(t) &= T(t - t_0)x_1 + \int_{t_0}^t T(t - s)[f(s, u(s), v(s)) \\ &\quad + \int_{t_0}^s k(s - \tau)g(\tau, u(\tau), v(\tau))d\tau]ds, \quad t_0 \leq t \leq t_1. \end{aligned} \quad (4.2.4)$$

4.3 Local Existence of Solutions

As we have already pointed, without loss of generality, that the semigroup generated by $-A$ can be assumed to be bounded and A is invertible. Under these conditions imposed on A, we prove the following local existence and uniqueness theorem.

Theorem 4.3.1 *Suppose that $-A$ generates the analytic semigroup $T(t)$ such that $\|T(t)\| \leq M$ and $0 \in \rho(-A)$. If the maps f and g satisfy Assumption (F) and Assumption (G), respectively, and kernel k satisfy (K), then (4.2.1) has a unique local classical solution.*

Proof. Fix (t_0, x_0, x_1) in U and choose $t_1' > t_0$ and $\delta > 0$ such that (4.2.2), with some fixed constant $L_f > 0$, $0 < \vartheta \leq 1$ and (4.2.3) with nonnegative function $L_g(t)$ hold on the set

$$V = \{(t, x, \tilde{x}) \in U \mid t_0 \leq t \leq t_1',\ \|x - x_0\|_1 + \|\tilde{x} - x_1\|_\alpha \ \leq\ \delta\}.$$

Let

$$B_f = \max_{t_0 \leq t \leq t_1'} \|f(t, x_0, x_1)\|,$$

$$B_g = \max_{t_0 \leq t \leq t_1'} \|g(t, x_0, x_1)\|$$

and

$$C(\delta) = [L_f + \|k\|_{L^p(t_0,t_1')}\|L_g\|_{L^q(t_0,t_1')}]\delta + B_f + B_g\|k\|_{L^p(t_0,t_1')}(t_1' - t_0)^{\frac{1}{q}}.$$

Choose $t_1 > t_0$ such that

$$\|T(t - t_0)x_1 - x_1\| + \|T(t - t_0)A^\alpha x_1 - A^\alpha x_1\| \ \leq\ \frac{\delta}{3}$$

and

$$t_1 - t_0 \ <\ \min\left\{t_1' - t_0, \frac{\delta}{3}(M + 1)^{-1}C(\delta)^{-1}, [\frac{\delta}{3}C_\alpha^{-1}(1 - \alpha)C(\delta)^{-1}]^{\frac{1}{1-\alpha}}\right\},$$

where C_α is a positive constant depending on α and satisfying

$$\|A^\alpha T(t)\| \;\leq\; C_\alpha t^{-\alpha} \quad \text{for} \quad t > 0. \tag{4.3.1}$$

Let $Y = C([t_0, t_1] : X \times X)$. Then, $y \in Y$ is of the form $y = (y_1, y_2)$, $y_i \in C([t_0, t_1] : X)$, $i = 1, 2$. Y, endowed with the supremum norm,

$$\|(y_1, y_2)\|_Y = \sup_{t_0 \leq t \leq t_1} [\|y_1(t)\| + \|y_2(t)\|],$$

is a Banach space. We define a map F on Y by $Fy = F(y_1, y_2) := (\hat{y}_1, \hat{y}_2)$ with

$$\begin{aligned} \hat{y}_1(t) &= Ax_0 - (T(t - t_0) - I)x_1 - \int_{t_0}^{t} (T(t - s) - I)F_y(s)ds, \\ \hat{y}_2(t) &= T(t - t_0)A^\alpha x_1 + \int_{t_0}^{t} T(t - s)A^\alpha F_y(s)ds, \end{aligned} \tag{4.3.2}$$

where

$$F_y(t) = f(t, A^{-1}y_1(t), A^{-\alpha}y_2(t)) + \int_{t_0}^{t} k(t - \tau)g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau))d\tau,$$

for $t \in [t_0, t]$.

For every $y \in Y$, $Fy(t_0) = (Ax_0, A^\alpha x_1)$. Now, we show that the map $t \mapsto Fy(t)$ is continuous function from $[t_0, t_1]$ to $X \times X$. For $s \leq t$,

$$\begin{aligned} Fy(t) - Fy(s) &= (\hat{y}_1(t), \hat{y}_2(t)) - (\hat{y}_1(s), \hat{y}_2(s)) \\ &= (\hat{y}_1(t) - \hat{y}_1(s), \hat{y}_2(t) - \hat{y}_2(s)). \end{aligned}$$

Taking norm on both side, we have

$$
\begin{aligned}
\|Fy(t)-Fy(s)\| &= \|\hat{y}_1(t)-\hat{y}_1(s)\|+\|\hat{y}_2(t)-\hat{y}_2(s))\| \\
&\leq \|T(t-t_0)-T(s-t_0)\|\|x_1\| \\
&\quad + \int_{t_0}^{s} \|T(t-\tau)-T(s-\tau)\|\|F_y(\tau)\|d\tau \\
&\quad + \int_{s}^{t} \|T(t-\tau)-I\|\|F_y(\tau)\|d\tau \\
&\quad + \|T(t-t_0)-T(s-t_0)\|\|A^\alpha x_1\| \\
&\quad + \int_{t_0}^{s} \|(T(t-\tau)-T(s-\tau))A^\alpha\|\|F_y(\tau)\|d\tau \\
&\quad + \int_{s}^{t} \|T(t-\tau)A^\alpha\|\|F_y(\tau)\|d\tau. \qquad (4.3.3)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\|f(t,A^{-1}y_1(t),A^{-\alpha}y_2(t))\| \\
&\quad \leq \|f(t,A^{-1}y_1(t),A^{-\alpha}y_2(t))-f(t,x_0,x_1)\|+\|f(t,x_0,x_1)\| \\
&\quad \leq L_f[\|y_1(t)-Ax_0\|+\|y_2(t)-A^\alpha x_1\|]+B_f \\
&\quad \leq L_f \sup_{t_0\leq t\leq t_1'} [\|y_1(t)-Ax_0\|+\|y_2(t)-A^\alpha x_1\|]+B_f.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\|g(t,A^{-1}y_1(t),A^{-\alpha}y_2(t))\| \\
&\quad \leq L_g(t) \sup_{t_0\leq t\leq t_1'} [\|y_1(t)-Ax_0\|+\|y_2(t)-A^\alpha x_1\|]+B_g.
\end{aligned}
$$

Therefore, for $F_y(t)$ we have the estimate

$$\begin{aligned}\|F_y(t)\| &= \|f(t, A^{-1}y_1(t), A^{-\alpha}y_2(t))\| \\ &\quad + \int_{t_0}^{t} |k(t-\tau)|\|g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau))\|d\tau \\ &\leq (L_f + \|k\|_{L^p(t_0,t_1')}\|L_g\|_{L^q(t_0,t_1')}) \sup_{t_0\leq t\leq t_1'} [\|y_1(t) - Ax_0\| + \|y_2(t) - A^\alpha x_1\|] \\ &\quad + (B_g + B_g\|k\|_{L^p(t_0,t_1')}(t_1' - t_0)^{1/q}) := N_1,\end{aligned}$$

since $y_1,\ y_2 \in C([t_0, t_1] : X)$. Thus, we have estimates for the second and the fourth integrals

$$\int_s^t \|T(t-\tau) - I\|\|F_y(\tau)\|d\tau \leq (M+1)N_1(t-s), \tag{4.3.4}$$

$$\int_s^t \|T(t-\tau)A^\alpha\|\|F_y(\tau)\|d\tau \leq C_\alpha N_1 \frac{(t-s)^{1-\alpha}}{1-\alpha}. \tag{4.3.5}$$

Part (d) of Theorem 1.3.19 implies that for $0 < \beta \leq 1$ and $x \in D(A^\beta)$, we have

$$\|(T(t) - I)x\| \leq C_\beta' t^\beta \|x\|_\beta. \tag{4.3.6}$$

If $0 < \beta < 1$ is such that $0 < \alpha + \beta < 1$, then $A^\alpha y \in D(A^\beta)$ for any $y \in D(A^{\alpha+\beta})$. Therefore, for $t, s \in [0, T]$, we have

$$\begin{aligned}\|(T(t) - I)A^\alpha T(s)x\| &\leq C_\beta' t^\beta \|A^\alpha T(s)x\|_\beta = C_\beta' t^\beta \|A^{\alpha+\beta}T(s)x\| \\ &\leq C_\beta' C_{\alpha+\beta} t^\beta s^{-(\alpha+\beta)}\|x\|.\end{aligned} \tag{4.3.7}$$

From (4.3.7), we get

$$\begin{aligned}\|(T(t-\tau) - T(s-\tau))A^\alpha\| &= \|(T(t-s) - I)A^\alpha T(t-\tau)\| \\ &\leq C_\beta' C_{\alpha+\beta}(t-s)^\beta (s-\tau)^{-(\alpha+\beta)}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{t_0}^{s} \|(T(t-\tau) &- T(s-\tau))A^\alpha\| \|F_y(\tau)\| d\tau \\ &\le N_1 C'_\beta C_{\alpha+\beta}(t-s)^\beta \frac{(t'_1-t_0)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}. \end{aligned} \tag{4.3.8}$$

Also, from (4.3.6), we have

$$\begin{aligned}\|(T(t)-I)T(s)x\| &\le C'_\beta t^\beta \|T(s)x\|_\beta = C'_\beta t^\beta \|A^\beta T(s)x\| \\ &\le C'_\beta C_\beta t^\beta s^{-\beta}\|x\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|T(t-\tau)-T(s-\tau)\| &= \|(T(t-s)-I)T(s-\tau)\| \\ &\le C'_\beta C_\beta (t-s)^\beta (s-\tau)^{-\beta}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{t_0}^{s} \|T(t-\tau) &- T(s-\tau)\| \|F_y(\tau)\| d\tau \\ &\le N_1 C'_\beta C_\beta (t-s)^\beta \frac{(t'_1-t_0)^{1-\beta}}{1-\beta}. \end{aligned} \tag{4.3.9}$$

Inequalities (4.3.3), (4.3.4), (4.3.5), (4.3.8) and (4.3.9) implies that $F : Y \to Y$. Let S be a nonempty closed and bounded set given by

$$S = \{y \in Y \mid y = (y_1, y_2), y_1(t_0) = Ax_0, y_2(t_0) = A^\alpha x_1, \|y_1(t)-Ax_0\|+\|y_2(t)-A^\alpha x_1\| \le \delta\}.$$

Let $y = (y_1, y_2)$ be any element of S. From (4.3.2), we have

$$\begin{aligned}&\|\hat{y}_1(t)-Ax_0\| + \|\hat{y}_2(t)-A^\alpha x_1\| \\ &\le \|(T(t-t_0)-I)x_1\| + \int_{t_0}^{t} \|T(t-s)-I\| \|F_y(s)\| ds \\ &\quad + \|(T(t-t_0)-I)A^\alpha x_1\| + \int_{t_0}^{t} \|A^\alpha T(t-s)\| \|F_y(s)\| ds. \end{aligned} \tag{4.3.10}$$

To find the estimate for $F_y(s)$, adding and subtracting $f(s,x_0,x_1)$ and $g(s,x_0,x_1)$ and using (F), (G) and (K), we obtain

$$\begin{aligned}\|F_y(s)\| &\le \|f(s,A^{-1}y_1(s),A^{-\alpha}y_2(s))-f(s,x_0,x_1)\|+B_f\\ &\quad+\int_{t_0}^{s}|k(s-\tau)|[\|g(\tau,A^{-1}y_1(\tau),A^{-\alpha}y_2(\tau))-g(\tau,x_0,x_1)\|+B_g]d\tau\\ &\le [L_f+\|k\|_{L^p(t_0,t_1')}\|L_g\|_{L^q(t_0,t_1')}]\delta+B_f+B_g\|k\|_{L^p(t_0,t_1')}(t_1'-t_0)^{\frac{1}{q}}\\ &\le C(\delta). \end{aligned}\tag{4.3.11}$$

Using estimate (4.3.11) and the fact that $\|T(t)\|\le M$ and (4.3.1), we get from (4.3.10)

$$\begin{aligned}\|\hat{y}_1(t)-Ax_0\| &+ \|\hat{y}_2(t)-A^{\alpha}x_1\|\\ &\le \frac{\delta}{3}+(M+1)C(\delta)(t-t_0)+\frac{C_\alpha C(\delta)(t-t_0)^{1-\alpha}}{1-\alpha}\\ &\le \delta.\end{aligned}$$

Hence, $F:S\to S$. Now, we show F is a contraction on S. Let (y_1,y_2) and (z_1,z_2) be any two point of S. From (4.3.2), we have

$$\begin{aligned}\|\hat{y}_1(t)-\hat{z}_1(t)\| &+ \|\hat{y}_2(t)-\hat{z}_2(t)\|\\ &\le \int_{t_0}^{t}\|T(t-s)-I\|\|F_y(s)-F_z(s)\|ds\\ &\quad+\int_{t_0}^{t}\|T(t-s)A^{\alpha}\|\|F_y(s)-F_z(s)\|ds.\end{aligned}\tag{4.3.12}$$

Using (F), (G) and (K), we get

$$\begin{aligned}\|F_y(s) &- F_z(s)\|\\ &\le \|f(s,A^{-1}y_1(s),A^{-\alpha}y_2(s))-f(s,A^{-1}z_1(s),A^{-\alpha}z_2(s))\|\\ &\quad+\int_{t_0}^{s}|a(s-\tau|\|g(\tau,A^{-1}y_1(\tau),A^{-\alpha}y_2(\tau))-g(\tau,A^{-1}z_1(\tau),A^{-\alpha}z_2(\tau))\|d\tau\\ &\le [L_f+\|k\|_{L^p(t_0,t_1')}\|L_g\|_{L^q(t_0,t_1')}]\|(y_1,y_2)-(z_1,z_2)\|_Y\\ &\le \frac{C(\delta)}{\delta}\|(y_1,y_2)-(z_1,z_2)\|_Y.\end{aligned}\tag{4.3.13}$$

Using (4.3.13) in (4.3.12), we get

$$\begin{aligned}\|\hat{y}_1(t)-\hat{z}_1(t)\| &+ \|\hat{y}_2(t)-\hat{z}_2(t)\| \\ &\le \left[\frac{(M+1)C(\delta)(t-t_0)}{\delta}+\frac{C_\alpha C(\delta)(t-t_0)^{1-\alpha}}{\delta(1-\alpha)}\right]\|(y_1,y_2)-(z_1,z_2)\|_Y \\ &\le \frac{2}{3}\|(y_1,y_2)-(z_1,z_2)\|_Y.\end{aligned}$$

Taking supremum over $[t_0,t_1]$, we have

$$\|(\hat{y}_1,\hat{y}_2)-(\hat{z}_1,\hat{z}_2)\|_Y \le \frac{2}{3}\|(y_1,y_2)-(z_1,z_2)\|_Y.$$

Thus, F is a contraction on S. Therefore, it has a unique fixed point in S. Let $\bar{y}=(\bar{y}_1,\bar{y}_2)\in S$ be that fixed point of F. Then,

$$\begin{aligned}\bar{y}_1(t) &= Ax_0-(T(t-t_0)-I)x_1-\int_{t_0}^t (T(t-s)-I)F_{\bar{y}}(s)ds, \\ \bar{y}_2(t) &= T(t-t_0)A^\alpha x_1+\int_{t_0}^t T(t-s)A^\alpha F_{\bar{y}}(s)ds, \qquad (4.3.14)\end{aligned}$$

where

$$F_{\bar{y}}(t)=f(t,A^{-1}\bar{y}_1(t),A^{-\alpha}\bar{y}_2(t))+\int_{t_0}^t k(t-\tau)g(\tau,A^{-1}\bar{y}_1(\tau),A^{-\alpha}\bar{y}_2(\tau))d\tau.$$

We note that $(u,v)=(A^{-1}\bar{y}_1,A^{-\alpha}\bar{y}_2)$ is the unique solution of the integral equations (4.2.4) on $[t_0,t_1]$. We can easily check that Assumption (F) and the continuity of $\bar{y}_1$ and $\bar{y}_2$ on $[t_0,t_1]$ imply that the map $t\mapsto F_{\bar{y}}(t)$ is continuous and hence bounded on $[t_0,t_1]$. Let $\|F_{\bar{y}}(t)\|\le N$ for $t_0\le t\le t_1$. We will now show that $t\mapsto F_{\bar{y}}(t)$ is locally Hölder continuous on $(t_0,t_1]$. For this, we first show that $\bar{y}_1$ and $\bar{y}_2$ are locally Hölder continuous on $(t_0,t_1]$. From Theorem 2.6.13 in Pazy [77], for every $0<\beta<1-\alpha$ and every $0<h<1$, we have

$$\begin{aligned}\|(T(h)-I)A^\alpha T(t-s)\| &\le C_\beta h^\beta\|A^{\alpha+\beta}T(t-s)\| \\ &\le Ch^\beta(t-s)^{-(\alpha+\beta)}. \qquad (4.3.15)\end{aligned}$$

Now

$$\begin{aligned}\|\bar{y}_2(t+h)-\bar{y}_2(t)\| &\leq \|(T(h)-I)A^\alpha T(t-t_0)x_1\| \\ &+\int_{t_0}^{t}\|(T(h)-I)A^\alpha T(t-s)F_{\bar{y}}(s)\|ds \\ &+\int_{t}^{t+h}\|A^\alpha T(t+h-s)F_{\bar{y}}(s)\|ds = I_1+I_2+I_3.\end{aligned}$$

We use (4.3.15) to get

$$\begin{aligned} I_1 &\leq C(t-t_0)^{-(\alpha+\beta)}h^\beta \leq M_1 h^\beta, \\ I_2 &\leq NCh^\beta\int_{t_0}^{t}(t-s)^{-(\alpha+\beta)}ds = \frac{NCh^\beta(t-t_0)^{1-(\alpha+\beta)}}{1-(\alpha+\beta)} \leq M_2h^\beta, \\ I_3 &\leq NC_\alpha\int_{t}^{t+h}(t+h-s)^{-\alpha} = \frac{NC_\alpha h^{1-\alpha}}{1-\alpha} \leq M_3h^\beta.\end{aligned}$$

Here, M_1 depends on t and increases to infinity as $t \downarrow t_0$, while M_2 and M_3 can be chosen independent of t. From the above estimates, it follows that there exists a positive constant C such that for every $t_0' > t_0$,

$$\|\bar{y}_2(t)-\bar{y}_2(s)\| \leq C|t-s|^\beta \quad \text{for} \quad t_0 < t_0' \leq t, \quad s \leq t_1.$$

Similar result holds for $\bar{y}_1$ (if we take $\alpha = 0$ in above consideration). For $s, t \in (t_0, t_1]$ with $t > s$, we have

$$\begin{aligned}\|F_{\bar{y}}(t)-F_{\bar{y}}(s)\| &\leq \|f(t, A^{-1}\bar{y}_1(t), A^{-\alpha}\bar{y}_2(t)) - f(s, A^{-1}\bar{y}_1(s), A^{-\alpha}\bar{y}_2(s))\| \\ &+\int_{t_0}^{s}|k(t-\tau)-a(s-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau \\ &+\int_{s}^{t}|k(t-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau.\end{aligned}$$

Since k is hölder continuous with exponent μ, we have

$$\begin{aligned}&\int_{t_0}^{s}|k(t-\tau)-k(s-\tau)|\|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\|d\tau \\ &\qquad \leq N(t_1-t_0)|t-s|^\mu \qquad (4.3.16)\end{aligned}$$

and

$$\int_s^t |k(t-\tau)|\|g(\tau, A^{-1}\bar{y_1}(\tau), A^{-\alpha}\bar{y_2}(\tau))\|d\tau \leq Nk_0(t_1-t_0)^\alpha|t-s|^{1-\alpha}, \quad (4.3.17)$$

where $k_0 = \max_{t_0\leq t\leq t_1}|k(t)|$. The local Hölder continuity of $F_{\bar{y}}(t)$ on $(t_0, t_1]$ follows from the Assumption (F), and the local Hölder continuity of $\bar{y_1}$ and $\bar{y_2}$ on $(t_0, t_1]$ and from estimates (4.3.16) and (4.3.17).

Consider the inhomogeneous initial value problem

$$\frac{dv(t)}{dt} + Av(t) = F_{\bar{y}}(t), \quad v(t_0) = x_1. \quad (4.3.18)$$

By the Corollary 1.3.20, this problem has a unique solution $v \in C^1((t_0, t_1] : X)$ given by

$$v(t) = T(t-t_0)x_1 + \int_{t_0}^t T(t-s)F_{\bar{y}}(s)ds, \quad (4.3.19)$$

for $t > t_0$. Each term on right hand side belongs to $D(A)$ and hence belongs to $D(A^\alpha)$ since $D(A) \subset D(A^\alpha)$, $0 \leq \alpha \leq 1$. Operating on both sides of (4.3.19) with A^α, we find

$$A^\alpha v(t) = T(t-t_0)A^\alpha x_1 + \int_{t_0}^t T(t-s)A^\alpha F_{\bar{y}}(s)ds, \quad (4.3.20)$$

By (4.3.14), the right hand side of (4.3.20) equals to $\bar{y_2}(t)$ and therefore $A^\alpha v(t) = \bar{y_2}(t)$ or $v(t) = A^{-\alpha}\bar{y_2}(t)$. Let $u(t) = A^{-1}\bar{y_1}(t)$, then we have $u(t) = x_0 + \int_{t_0}^t v(s)ds$, which yields $u(t) \in C^1([t_0, t_1) : X) \cap C^2((t_0, t_1) : X)$. Thus, u satisfies (4.2.1) on $[t_0, t_1)$. This completes the proof of Theorem 4.3.1.

4.4 Global Existence of Solutions

In this section, we will prove under additional growth condition on the nonlinear map f and g the following global existence result.

Theorem 4.4.1 *Let $0 \in D(-A)$ and $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ such that $\|T(t)\| \leq M$ for $t \geq 0$. Let $f, g : [0, \infty) \times X_1 \times X_\alpha \mapsto X$ satisfy Assumption (F) and Assumption (G), respectively, and k satisfies (K). If there exists a nondecreasing function $a_f : [t_0, \infty) \mapsto R_+$ and a nonnegative function $a_g \in L^q_{loc}(0, \infty)$ where q is same as before, such that*

$$\begin{aligned} \|f(t,x,\tilde{x})\| &\leq a_f(t)[1 + \|x\|_1 + \|\tilde{x}\|_\alpha] \quad \text{for} \quad t \geq t_0, (x, \tilde{x}) \in X_1 \times X_\alpha, \\ \|g(t,x,\tilde{x})\| &\leq a_g(t)[1 + \|x\|_1 + \|\tilde{x}\|_\alpha] \quad \text{for} \quad t \geq t_0, (x, \tilde{x}) \in X_1 \times X_\alpha, \end{aligned}$$

then for each $(x_0, x_1) \in X_1 \times X_\alpha$, (4.2.1) has a unique classical solution u which exists for all $t \geq t_0$.

For the proof of this theorem we shall require the following lemma.

Lemma 4.4.2 *Let $\phi(t,s) \geq 0$ be continuous on $0 \leq s < t \leq T$. If there are positive constants A, B_1, B_2 and β, such that*

$$\phi(t,s) \leq A + B_1 \int_s^t \phi(\sigma, s) d\sigma + B_2 \int_s^t (t-\sigma)^{\beta-1} \phi(\sigma, s) d\sigma, \tag{4.4.1}$$

for $0 \leq s < t \leq T$, then there exists a positive constant C such that $\phi(t,s) \leq C$ for $0 \leq s < t \leq T$.

Proof. We have

$$\int_s^t \int_s^\sigma (t-\sigma)^{\beta-1} \phi(\tau, s) d\tau d\sigma = \int_s^t \left(\int_\tau^t (t-\sigma)^{\beta-1} d\sigma \right) \phi(\tau, s) d\tau \tag{4.4.2}$$

and a well-known identity

$$\int_\tau^t (t-\tau)^{\beta-1}(\sigma-\tau)^{\gamma-1}d\sigma = (t-\tau)^{\beta+\gamma-1}\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}. \tag{4.4.3}$$

Iterating (4.4.1) $n-1$ times, using (4.4.2) and (4.4.3) and majorating $(t-s)$ and $(t-\tau)$ by T, we get

$$\begin{aligned}\phi(t,s) \ &\leq \ A\sum_{j=0}^{n-1}\left(\frac{B_2T^\beta}{\beta}\right)^j + B_1\sum_{j=0}^{n-1}\left(\frac{B_2T^\beta}{\beta}\right)^j\int_s^t \phi(\sigma,s)d\sigma \\ &\quad + \frac{(B_2\Gamma(\beta))^n}{\Gamma(n\beta)}\int_s^t (t-\sigma)^{n\beta-1}\phi(\sigma,s)d\sigma.\end{aligned}$$

Choosing n sufficiently large so that $n\beta > 1$ and replacing $(t-\sigma)^{n\beta-1}$ by $T^{n\beta-1}$, we get

$$\phi(t,s) \ \leq \ C_1 + C_2\int_s^t \phi(\sigma,s)d\sigma,$$

where C_1 and C_2 are the positive constants independent of s. The required result then follows from Gronwall's inequality.

Proof of Theorem 4.4.1. Let $[t_0,T)$ be the maximal interval of existence for the solution u to (4.2.1) guaranteed by Theorem 4.3.1. It is suffices to prove that

$$\|u(t)\|_1 + \|v(t)\|_\alpha \ \leq \ C$$

on $[t_0,T)$ for some fixed constant $C \geq 0$ independent of t.

Now, since $u(t)$ is a solution of (4.2.1) on $[t_0,T)$, it is also a mild solution to (4.2.1) therefore from (4.3.14), we have

$$\begin{aligned} Au(t) &= Ax_0 - (T(t-t_0)-I)x_1 - \int_{t_0}^t (T(t-s)-I)\bar{F}(s)ds, \\ A^\alpha u'(t) &= T(t-t_0)A^\alpha x_1 + \int_{t_0}^t T(t-s)A^\alpha \bar{F}(s)ds, \end{aligned} \tag{4.4.4}$$

where

$$\bar{F}(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t-\tau) g(\tau, u(\tau), u'(\tau)) d\tau.$$

From (4.4.4), we have

$$\begin{aligned} [1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] &= [1 + \|Au(\eta)\| + \|A^\alpha u'(\eta)\|] \\ &\leq 1 + \|Ax_0\| + (M+1)\|x_1\| + (M+1) \int_{t_0}^{\eta} \|\bar{F}(s)\| ds \\ &\quad + M \|x_1\|_\alpha + \int_{t_0}^{\eta} C_\alpha (\eta - s)^{-\alpha} \|\bar{F}(s)\| ds. \end{aligned} \tag{4.4.5}$$

Our assumptions on f, g and k imply that

$$\begin{aligned} \|\bar{F}(s)\| &\leq \|f(t, u(t), u'(t))\| + \int_{t_0}^{s} |k(s-\tau)| \|g(\tau, u(\tau), u'(\tau))\| d\tau \\ &\leq (a_f(T) + \|k\|_{L^p(t_0,T)} \|a_g\|_{L^q(t_0,T)}) \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha]. \end{aligned} \tag{4.4.6}$$

Using (4.4.6) in (4.4.5), we get

$$\begin{aligned} [1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] &\leq C_1 + C_2 \int_{t_0}^{\eta} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds \\ &\quad + C_3 \int_{t_0}^{\eta} (\eta - s)^{-\alpha} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds. \end{aligned}$$

Hence, we have

$$\begin{aligned} \sup_{t_0 \leq \eta \leq s} [1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] &\leq C_1 + C_2 \int_{t_0}^{\eta} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds \\ &\quad + C_3 \int_{t_0}^{\eta} (\eta - s)^{-\alpha} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds. \end{aligned}$$

Using Lemma 4.4.2, we have

$$\sup_{t_0 \leq \eta \leq s} [1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] \leq C.$$

Hence, we get the required result. This completes the proof of the theorem.

Chapter 5

Second Order Semi-linear Integro-differential Equations - II

I have tried to avoid long numerical computations, thereby following Riemann's postulate that proofs should be given through idea and not voluminous computations.

— David Hilbert

5.1 Introduction

In this chapter, we shall continue the study of a second order semi-linear integro-differential equation. In chapter 4, using the semigroup theory and the contraction mapping principle, we have proved the existence of a unique local classical solution of the abstract second order semi-linear integro-differential equation

$$\begin{aligned}
&u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t-s)g(s, u(s), u'(s))ds, \quad t > t_0, \\
&u(t_0) = x_0, \quad u'(t_0) = x_1,
\end{aligned}$$

in a Banach space and discussed the continuation of this solution, the maximal interval of existence and the global existence under additional growth conditions on the nonlinear maps f and g. In this chapter we establish the results for the approximation to the solution of this equation in a Hilbert space.

Let Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Consider the following initial boundary value problem for the strongly damped semi-linear integro-differential equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) + (aL+bI)\left(\frac{\partial u}{\partial t}\right)(x,t) \ + \ (cL+dI)u(x,t) \ &= \ h(x,t,u(x,t),\frac{\partial u}{\partial t}(x,t)) \\ + \int_{t_0}^{t} k(t-s)g(x,s,u(x,s),\frac{\partial u}{\partial s}(x,s))ds, \quad (x,t) &\in \Omega\times(0,T), \\ u(x,0)=x_0(x), \quad \frac{\partial u}{\partial t}(x,0)=x_1(x), \quad x\in\Omega, & \qquad (5.1.1) \end{aligned}$$

with the homogeneous Dirichlet boundary conditions, where $Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ be a symmetric second order strongly elliptic differential operator in Ω, $a>0$, b, c, d are constants and h and g are smooth nonlinear functions and k is a locally p-integrable function for $1<p<\infty$.

In the abstract form, we may write (5.1.1) as the initial value problem

$$\begin{aligned} &\frac{d^2u}{dt^2}(t) + A\left(\frac{du}{dt}\right)(t) + Bu(t) \\ &\quad = f(t,u(t),\frac{du}{dt}(t)) + \int_{t_0}^{t} k(t-s)g(s,u(s),\frac{du}{ds}(s))ds, \quad t>t_0, \\ &u(0)=x_0, \quad \frac{du}{dt}(0)=x_1, \qquad (5.1.2) \end{aligned}$$

in the separable Hilbert space $H=L^2(\Omega)$, where the linear operator A with the domain $D(A)=H^2(\Omega)\cap H_0^1(\Omega)$ is given by

$$Au = aLu, \quad u\in D(A),$$

and the operator B is such that $D(B) = D(A)$ with $Bu = c_1A + c_2I$ for some constant c_1 and c_2. The function f is defined from $[0, \infty) \times H \times H$ into H given by

$$f(t, u, v) = h(t, u, v) - bv.$$

In this chapter again we are concerned with the convergence of the Faedo-Galerkin approximation of the solution to (5.1.2). Without loss of generality, we concentrate on the abstract second order semi-linear integro-differential equations of the form

$$\begin{aligned} &u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_0^t k(t-s)g(s, u(s), u'(s))ds, \\ &u(0) = x_0, \quad u'(0) = x_1, \end{aligned} \tag{5.1.3}$$

in a separable Hilbert space $(H, \|.\|, (.,.))$, since the other terms may be transferred on the right hand side and merged with f as the condition satisfied by the changed f still remains the same. We extend the technique used in Chapter 3 for a class of second order semi-linear evolution equations to integro-differential equation (5.1.3).

The organization of this chapter is as follows. In the second section we state some preliminaries and the assumptions required for establishing results. In the third section we consider a pair of associated nonlinear integral equations to (5.1.3). The solutions to these integral equations are defined as mild solutions to (5.1.3). With the help of the pair of associated nonlinear integral equations and the projection operators, we consider a pair of approximate nonlinear integral equations and show the existence and uniqueness of a solution to this pair of approximate integral equations using the Banach contraction principle in this section. In the fourth section we establish the convergence of solutions and convergence of the pair of approximate integral equations to the pair of associated integral equations, limit of the solutions to the approximate integral equations being the solution of the pair of associated integral equations. Further, we show in this section that the solution can be extended to the maximal interval of existence and it is unique.

Finally, in the fifth section we consider the Faedo-Galerkin approximations of solutions and prove some results concerning the convergence of such approximations.

5.2 Preliminaries and Assumptions

In this section, we consider some preliminaries and assumptions essential for our purpose. The operator A assume to satisfy the following assumptions:

(H1) A is a closed, positive definite, self-adjoint, linear operator from the domain $D(A) \subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq .....$$

and a corresponding complete orthonormal system of eigen functions $\{u_i\}$, i.e.,

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If (H1) is satisfied, then $-A$ generates an analytic semigroup in H which we denote by e^{-tA}, $t \geq 0$.

Now, we mention some notions and preliminaries. It is well known that there exists constant $\tilde{M} \geq 1$ and a real number ω such that

$$\|e^{-tA}\| \leq \tilde{M} e^{\omega t}, \quad t \geq 0.$$

Since $-A$ generates the analytic semigroup e^{-tA}, $t \geq 0$, we may add cI to $-A$ for some constant c, if necessary, and in what follows we may assume without loss of generality that $\|e^{-tA}\|$ is uniformly bounded by M, i.e., $\|e^{-tA}\| \leq M$ and $0 \in \rho(A)$. In this case, it is possible to define the fractional power A^{α} for $0 < \alpha < 1$ as closed linear operator

with domain $D(A^\alpha) \subseteq H$ (cf. Pazy [77], pp. 69-75 and p. 195). Furthermore, $D(A^\alpha)$ is dense in H and the expression

$$\|x\|_\alpha = \|A^\alpha x\|,$$

defines a norm on $D(A^\alpha)$. Henceforth, we represent by X_α, the space $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$. In view of the facts mentioned above, we have the following result for an analytic semigroup e^{-tA}, $t \geq 0$ (cf. Pazy [77] pp. 195-196).

Lemma 5.2.1 *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup e^{-tA}, $t \geq 0$ with $\|e^{-tA}\| \leq M$ for $t \geq 0$ and $0 \in \rho(A)$. Then we have the following properties.*

(i) X_α for $0 \leq \alpha \leq 1$ is a Banach space.

(ii) For $0 < \beta \leq \alpha$, the embedding $X_\alpha \hookrightarrow X_\beta$ is continuous.

(iii) A^α commutes with e^{-tA} and there exists a constant $C_\alpha > 0$ depending on α such that

$$\|A^\alpha e^{-tA}\| \leq C_\alpha t^{-\alpha}, \quad t > 0.$$

The kernel k and the nonlinear maps f and g are assumed to satisfy the following assumptions.

(H2) $k \in L^p_{loc}(0, \infty)$ for some $1 < p < \infty$ be a bounded function.

(H3) The map f is defined from $[0, \infty) \times X_1 \times X_\alpha$ into H and there exists a nondecreasing function F_R from $[0, \infty)$ into $[0, \infty)$ depending on $R > 0$ such that

$$\|f(t, u, v)\| \leq F_R(t),$$

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq F_R(t)\{\|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha\},$$

for all (t, u, v) and (t, u_1, v_1), (t, u_2, v_2) in $[0, \infty) \times B_R(X_1 \times X_\alpha, (x_0, x_1))$, where

$$B_R(X_1 \times X_\alpha, (x_0, x_1)) = \{(x, y) \in X_1 \times X_\alpha \ : \ \|x - x_0\|_1 + \|y - x_1\|_\alpha \ \leq \ R\}.$$

(H4) The map g is defined from $[0, \infty) \times X_1 \times X_\alpha$ into H and there exists a nonnegative bounded function $G_R \in L^q_{loc}(0, \infty)$ depending on $R > 0$, where $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\begin{aligned}
\|g(t, u, v)\| \ &\leq \ G_R(t), \\
\|g(t, u_1, v_1) - g(t, u_2, v_2)\| \ &\leq \ G_R(t)\{\|u_1 - u_2\|_1 + \|v_1 - v_2\|_\alpha\},
\end{aligned}$$

for a.e. $t \in [0, \infty)$ and all (u, v), (u_1, v_1) and (u_2, v_2) in $B_R(X_1 \times X_\alpha, (x_0, x_1))$.

5.3 Approximate Integral Equations

we continue to use the notions and notations of the earlier section. The existence of a solution to (5.1.3) is closely associated with the following pair of integral equations

$$\begin{aligned}
u(t) &= x_0 + (e^{-tA} - I)(-A)^{-1}x_1 + \int_0^t (e^{-(t-s)A} - I)(-A)^{-1}[f(s, u(s), v(s)) \\
&\quad + \int_0^s k(s - \tau)g(\tau, u(\tau), v(\tau))d\tau]ds, \\
v(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f(s, u(s), v(s)) + \int_0^s k(s - \tau)g(\tau, u(\tau), v(\tau))d\tau]ds,
\end{aligned} \tag{5.3.1}$$

for $t \geq 0$. In this section, we shall consider a pair of approximate integral equations associated to (5.3.1) and establish the existence and uniqueness of a solution to the pair of approximate integral equations. By a solution (u, v) to (5.3.1) on $[0, T]$, $0 < T < \infty$, we mean a function $(u, v) \in X_1(T) \times X_\alpha(T)$ for some $0 < \alpha < 1$ satisfying (5.3.1) where $X_1(T) \times X_\alpha(T)$ is the Banach space $C([0, T], D(A) \times D(A^\alpha))$ of all continuous functions

from $[0,T]$ into $D(A)\times D(A^{\alpha})$ endowed with the supremum norm

$$\|(u,v)\|_{X_1(T)\times X_\alpha(T)} = \|u\|_{X_1(T)} + \|v\|_{X_\alpha(T)},$$

where

$$\|u\|_{X_1(T)} = \sup_{0\le t\le T}\|Au(t)\| = \sup_{0\le t\le T}\|u(t)\|_1$$

and

$$\|v\|_{X_\alpha(T)} = \sup_{0\le t\le T}\|A^{\alpha}v(t)\| = \sup_{0\le t\le T}\|v(t)\|_\alpha.$$

By a solution (u,v) to (5.3.1) on $[0,\tilde{T})$, $0<\tilde{T}\le\infty$, we mean a function $(u,v)\in X_1(T)\times X_\alpha(T)$ for some $0<\alpha<1$, satisfying (5.3.1) in $[0,T]$ for every $0<T<\tilde{T}$.

Let $0<T_0<\infty$ be arbitrary fixed and

$$L(R) = (1+R)[F_{\tilde{R}}(T_0) + \|k\|_{L^p(0,T_0)}\|G_{\tilde{R}}\|_{L^q(0,T_0)}],$$

where

$$\tilde{R} = \tilde{R}_1 + \tilde{R}_2, \quad \tilde{R}_1 = \sqrt{R^2+\|x_0\|_1^2} \ \text{ and } \tilde{R}_2 = \sqrt{R^2+\|x_1\|_\alpha^2}.$$

Let $0<T\le T_0$ be such that

$$\sup_{0\le t\le T}\{\|(e^{-tA}-I)x_1\| + \|(e^{-tA}-I)A^{\alpha}x_1\|\} \ < \ \frac{R}{3}$$

and

$$T \ < \ \min\left\{T_0, \frac{R}{3}(M+1)^{-1}L(R)^{-1}, [\frac{R}{3}C_\alpha^{-1}(1-\alpha)L(R)^{-1}]^{\frac{1}{1-\alpha}}\right\}.$$

Let H_n denote the finite dimensional subspace of H spanned by $\{u_0,u_1,\ldots,u_n\}$ and for $n=1,2,\ldots$; let $P^n : H\to H_n$ be the corresponding projection operator. For each n, we define

$$f_n,\ g_n : [0,T]\times X_1(T)\times X_\alpha(T)\to H,$$

such that

$$f_n(t,u,v,) = f(t,P^n u(t),P^n v(t)),$$

$$g_n(t,u,v,) = g(t,P^n u(t),P^n v(t)).$$

We set $\tilde{x_0}(t) \equiv x_0$ and $\tilde{x_1}(t) \equiv x_1$ for $t \in [0,T]$. Let $W_R = B_R(X_1(T) \times X_\alpha(T), (\tilde{x_0}, \tilde{x_1}))$, where

$$\begin{aligned} & B_R(X_1(T) \times X_\alpha(T), (\tilde{x_0}, \tilde{x_1})) \\ & \quad = \{(y_1, y_2) \in X_1(T) \times X_\alpha(T) \ : \ \|y_1 - \tilde{x_0}\|_{X_1(T)} + \|y_2 - \tilde{x_1}\|_{X_\alpha(T)} \ \le \ R\}. \end{aligned}$$

Define a map S_n on W_R such that $S_n(u,v) := (\hat{u}, \hat{v})$ with

$$\begin{aligned} \hat{u}(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}[f_n(s,u,v) \\ &\quad + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau]ds, \\ \hat{v}(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau]ds. \end{aligned} \tag{5.3.2}$$

Propostion 5.3.1 *Let (H1)-(H4) hold. Then there exists unique $(u_n, v_n) \in W_R$ such that $S_n(u_n, v_n) = (u_n, v_n)$ for each $n = 1, 2, \ldots$, i.e., (u_n, v_n) satisfies the pair of approximate integral equations*

$$\begin{aligned} u_n(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}[f_n(s,u_n,v_n) \\ &\quad + \int_0^s k(s-\tau)g_n(\tau,u_n,v_n)d\tau]ds, \\ v_n(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f_n(s,u_n,v_n) + \int_0^s k(s-\tau)g_n(\tau,u_n,v_n)d\tau]ds. \end{aligned} \tag{5.3.3}$$

Proof. First, we claim that $S_n : W_R \to W_R$. For this we need to show first that the map $t \mapsto (S_n(u,v))(t)$ is continuous from $[0,T]$ into $X_1 \times X_\alpha$ with respect to norm

$\|.\|_1 + \|.\|_\alpha$. For t_1, $t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\|\hat{u}(t_2) - \hat{u}(t_1)\|_1 + \|\hat{v}(t_2) - \hat{v}(t_1)\|_\alpha \\
\leq\ & \|(e^{-t_2 A} - e^{-t_1 A})x_1\| + \|(e^{-t_2 A} - e^{-t_1 A})x_1\|_\alpha \\
&+ \int_{t_1}^{t_2} \|e^{-(t_2-s)A} - I\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\
&+ \int_0^{t_1} \|e^{-(t_2-s)A} - e^{-(t_1-s)A}\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\
&+ \int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\
&+ \int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds.
\end{aligned} \tag{5.3.4}$$

Now, we estimate

$$\begin{aligned}
&\left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| \\
&\qquad \leq\ \|f_n(s,u,v)\| + \int_0^s |k(s-\tau)| \|g_n(\tau,u,v)\| d\tau \\
&\qquad \leq\ F_{\tilde{R}}(T_0) + \int_0^s |k(s-\tau)| G_{\tilde{R}}(\tau) d\tau \\
&\qquad \leq\ F_{\tilde{R}}(T_0) + \|k\|_{L^p(0,T_0)} \|G_{\tilde{R}}\|_{L^q(0,t_0)} \ \leq\ L(R).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \|e^{-(t_2-s)A} - I\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\
&\qquad \leq\ (M+1)L(R)(t_2 - t_1),
\end{aligned} \tag{5.3.5}$$

$$\begin{aligned}
&\int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\
&\qquad \leq\ L(R)C_\alpha \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} ds \\
&\qquad =\ L(R)C_\alpha \frac{(t_2-t_1)^{1-\alpha}}{1-\alpha}.
\end{aligned} \tag{5.3.6}$$

Part (d) of Theorem 1.3.19 implies that for $0 < \vartheta \leq 1$ and $x \in D(A^\vartheta)$, we have

$$\|(e^{-tA} - I)x\| \leq C'_\vartheta t^\vartheta \|x\|_\vartheta. \tag{5.3.7}$$

If $0 < \vartheta < 1$ is such that $0 < \alpha + \vartheta < 1$, then $A^\alpha y \in D(A^\vartheta)$ for any $y \in D(A^{\alpha+\vartheta})$. Therefore, for $t, s \in [0, T]$, we have

$$\begin{aligned} \|(e^{-tA} - I)A^\alpha e^{-sA}x\| &\leq C'_\vartheta t^\vartheta \|A^\alpha e^{-sA}x\|_\vartheta = C'_\vartheta t^\vartheta \|A^{\alpha+\vartheta} e^{-sA}x\| \\ &\leq C'_\vartheta C_{\alpha+\vartheta} t^\vartheta s^{-(\alpha+\vartheta)} \|x\|. \end{aligned} \tag{5.3.8}$$

From (5.3.8), we get

$$\begin{aligned} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| &= \|(e^{-(t_2-t_1)A} - I)A^\alpha e^{-(t_1-s)A}\| \\ &\leq C'_\vartheta C_{\alpha+\vartheta}(t_2 - t_1)^\vartheta (t_1 - s)^{-(\alpha+\vartheta)}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{t_1} \|(e^{-(t_2-s)A} &- e^{-(t_1-s)A})A^\alpha\| \left\| f_n(s, u, v) + \int_0^s k(s - \tau) g_n(\tau, u, v) d\tau \right\| ds \\ &\leq C'_\vartheta C_{\alpha+\vartheta} L(R)(t_2 - t_1)^\vartheta \int_0^{t_1} (t_1 - s)^{-(\alpha+\vartheta)} ds \\ &\leq C'_\vartheta C_{\alpha+\vartheta} L(R) \frac{T_0^{1-(\alpha+\vartheta)}}{1 - (\alpha+\vartheta)} (t_2 - t_1)^\vartheta. \end{aligned} \tag{5.3.9}$$

Also, from (5.3.7), we have

$$\begin{aligned} \|(e^{-tA} - I)e^{-sA}x\| &\leq C'_\vartheta t^\vartheta \|e^{-sA}x\|_\vartheta = C'_\vartheta t^\vartheta \|A^\vartheta e^{-sA}x\| \\ &\leq C'_\vartheta C_\vartheta t^\vartheta s^{-\vartheta} \|x\|. \end{aligned}$$

Therefore

$$\begin{aligned} \|e^{-(t_2-s)A} - e^{-(t_1-s)A}\| &= \|(e^{-(t_2-t_1)A} - I)e^{-(t_1-s)A}\| \\ &\le C'_\vartheta C_\vartheta (t_2-t_1)^\vartheta (t_1-s)^{-\vartheta}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{t_1} \|e^{-(t_2-s)A} &- e^{-(t_1-s)}\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\ &\le C'_\vartheta C_\vartheta L(R)(t_2-t_1)^\vartheta \int_0^{t_1} (t_1-s)^{-\vartheta} ds \\ &\le C'_\vartheta C_\vartheta L(R) \frac{T_0^{1-\vartheta}}{1-\vartheta}(t_2-t_1)^\vartheta. \end{aligned} \tag{5.3.10}$$

From inequalities (5.3.4), (5.3.5), (5.3.6), (5.3.9) and (5.3.10), it follows that $(S_n(u,v))(t)$ is continuous from $[0,T]$ into $D(A) \times D(A^\alpha)$ with respect to norm $\|.\|_1 + \|.\|_\alpha$. Now, we show that $S_n(u,v) \in W_R$ i.e., $(\hat{u},\hat{v}) \in W_R$. Consider

$$\begin{aligned} \|\hat{u}(t) - x_0\|_1 &+ \|\hat{v}(t) - x_1\|_\alpha \\ &\le \|(e^{-tA} - I)x_1\| + \|(e^{-tA} - I)A^\alpha x_1\| \\ &\quad + \int_0^t \|e^{-(t-s)A} - I\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\ &\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\| \left\| f_n(s,u,v) + \int_0^s k(s-\tau)g_n(\tau,u,v)d\tau \right\| ds \\ &\le \frac{R}{3} + (M+1)L(R)T + C_\alpha L(R) \int_0^t (t-s)^{-\alpha} ds \\ &\le \frac{R}{3} + (M+1)L(R)T + C_\alpha L(R) \frac{T^{1-\alpha}}{1-\alpha} \le R. \end{aligned}$$

Taking supremum over $[0,T]$, we get

$$\|\hat{u} - \tilde{x}_0\|_{X_1(T)} + \|\hat{v} - \tilde{x}_1\|_{X\alpha(T)} \le R,$$

which implies that $S_n(u,v) \in W_R$. So, S_n maps W_R into W_R. Now, it remains to show that S_n is contraction on W_R. For $(u_1,v_1),(u_2,v_2) \in W_R$, we have

$$\begin{aligned}
\|\hat{u}_1(t)-\hat{u}_2(t)\|_1 &+ \|\hat{v}_1(t)-\hat{v}_2(t)\|_\alpha \\
&\leq \int_0^t \|e^{-(t-s)A}-I\|\{\|f_n(s,u_1,v_1)-f_n(s,u_2,v_2)\| \\
&\quad + \int_0^s |k(s-\tau)|\|g_n(\tau,u_1,v_1)-g_n(\tau,u_2,v_2)\|d\tau\}ds \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\|\{\|f_n(s,u_1,v_1)-f_n(s,u_2,v_2)\| \\
&\quad + \int_0^s |k(s-\tau)|\|g_n(\tau,u_1,v_1)-g_n(\tau,u_2,v_2)\|d\tau\}ds.
\end{aligned} \tag{5.3.11}$$

From (H3), we have

$$\begin{aligned}
\|f_n(s,u_1,v_1) &- f_n(s,u_2,v_2)\| \\
&= \|f(s,P^nu_1(s),P^nv_1(s))-f(s,P^nu_2(s),P^nv_2(s))\| \\
&\leq F_{\tilde{R}}(T_0)(\|P^nu_1(s)-P^nu_2(s)\|_1+\|P^nv_1(s)-P^nv_2(s)\|_\alpha) \\
&\leq F_{\tilde{R}}(T_0)(\|u_1(s)-u_2(s)\|_1+\|v_1(s)-v_2(s)\|_\alpha) \\
&\leq F_{\tilde{R}}(T_0)(\|u_1-u_2\|_{X_1(T)}+\|v_1-v_2\|_{X_\alpha(T)}).
\end{aligned}$$

Similarly, from (H4), we get

$$\begin{aligned}
\|g_n(s,u_1,v_1) &- g_n(s,u_2,v_2)\| \\
&\leq G_{\tilde{R}}(s)(\|u_1-u_2\|_{X_1(T)}+\|v_1-v_2\|_{X_\alpha(T)}).
\end{aligned}$$

Therefore

$$
\begin{aligned}
\|f_n(s,u_1,v_1) - f_n(s,u_2,v_2)\| &+ \int_0^s |k(s-\tau)|\|g_n(\tau,u_1,v_1) - g_n(\tau,u_2,v_2)\|d\tau \\
&\leq [F_{\tilde{R}}(T_0) + \|k\|_{L^p(0,T_0)}\|G_{\tilde{R}}\|_{L^q(0,T_0)}](\|u_1-u_2\|_{X_1(T)} + \|v_1-v_2\|_{X_\alpha(T)}) \\
&\leq \frac{L(R)}{R}(\|u_1-u_2\|_{X_1(T)} + \|v_1-v_2\|_{X_\alpha(T)}). \qquad (5.3.12)
\end{aligned}
$$

Hence, using the estimate (5.3.12) in (5.3.11), we get

$$
\begin{aligned}
\|\hat{u}_1(t) - \hat{u}_2(t)\|_1 &+ \|\hat{v}_1(t) - \hat{v}_2(t)\|_\alpha \\
&\leq \frac{1}{R}\left\{(M+1)L(R)T + C_\alpha L(R)(T_0)\int_0^t (t-s)^{-\alpha}ds\right\} \\
&\quad \times (\|u_1-u_2\|_{X_1(T)} + \|v_1-v_2\|_{X_\alpha(T)}) \\
&\leq \frac{1}{R}\left\{(M+1)L(R)T + C_\alpha L(R)\frac{T^{1-\alpha}}{1-\alpha}\right\} \\
&\quad \times (\|u_1-u_2\|_{X_1(T)} + \|v_1-v_2\|_{X_\alpha(T)}) \\
&\leq \frac{2}{3}(\|u_1-u_2\|_{X_1(T)} + \|v_1-v_2\|_{X_\alpha(T)}).
\end{aligned}
$$

Taking supremum over $[0,T]$, we get

$$
\|\hat{u}_1 - \hat{u}_2\|_{X_1(T)} + \|\hat{v}_1 - \hat{v}_2\|_{X_\alpha(T)} \leq \frac{2}{3}(\|u_1-u_2\|_{X_1(T)} + \|v_1-v_2\|_{X_\alpha(T)}).
$$

Thus, S_n is a strict contraction on W_R. Hence, there exists a unique $(u_n, v_n) \in W_R$ such that $S_n(u_n,v_n) = (u_n,v_n)$. Clearly, (u_n,v_n) satisfies (5.3.3). This completes the proof of the Proposition.

Propostion 5.3.2 *Let (H1)-(H4) hold. If* $(x_0,x_1) \in D(A)\times D(A)$, *then* $(u_n(t), v_n(t)) \in D(A)\times D(A^\vartheta)$ *for all* $t \in [0,T]$ *where* $0 \leq \vartheta < 1$.

Proof. From Proposition 5.3.2 we have the existence of a unique $(u_n,v_n) \in B_R(X_1(T) \times X_\alpha(T), (\tilde{x}_0, \tilde{x}_1)$ satisfying (5.3.3). By Theorem 1.3.9, we have for $x \in H$, $\int_0^t e^{-tA}xds \in$

$D(A)$ and if $x \in D(A)$ then $e^{-tA}x \in D(A)$. The result of the proposition follows from these facts and the fact $D(A) \subseteq D(A^\vartheta)$ for $0 \leq \vartheta \leq 1$.

Propostion 5.3.3 *Let (H1)-(H4) hold. Then for $x_1 \in D(A)$ there exists constant V_0 independent of n, such that*

$$\|v_n(t)\|_\vartheta \;\leq\; V_0, \quad 0 \leq \vartheta < 1, \quad 0 \leq t \leq T.$$

Proof. Applying A^ϑ on both the sides of second integral equation of (5.3.3) and taking norm, for $0 \leq t \leq T$, we have

$$\begin{aligned}\|v_n(t)\|_\vartheta \;&\leq\; \|A^\vartheta e^{-tA}x_1\| + \int_o^t \|e^{-(t-s)A}A^\vartheta\|[\|f_n(s,u_n,v_n)\| \\ &+ \int_0^s |k(s-\tau)|\|g_n(\tau,u_n,v_n\|d\tau]ds.\end{aligned}$$

Since $x_1 \in D(A)$ implies that $x_1 \in D(A^\vartheta)$ for $0 \leq \vartheta < 1$, we get

$$\|v_n(t)\|_\vartheta \;\leq\; M\|x_1\|_\vartheta + C_\vartheta L(R)\frac{T^{1-\vartheta}}{1-\vartheta} \;\leq\; V_0.$$

This completes the proof of the proposition.

5.4 Convergence of Approximate Solutions

In this section, we establish the convergence of the solution $(u_n, v_n) \in X_1(T) \times X_\alpha(T)$ of the pair of approximate integral equations

$$\begin{aligned}u_n(t) \;&=\; x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}[f_n(s,u_n,v_n) \\ &+ \int_0^s k(s-\tau)g_n(\tau,u_n,v_n)d\tau]ds, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.4.1) \\ v_n(t) \;&=\; e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f_n(s,u_n,v_n) + \int_0^s k(s-\tau)g_n(\tau,u_n,v_n)d\tau]ds,\end{aligned}$$

then there exists a constant K such that

$$f(t)+\int_0^t f(\tau)d\tau \;\le\; AK \quad for \quad 0\le t\le T.$$

Furthermore,

$$f(t) \;\le\; AK \quad for \quad 0\le t\le T.$$

Proof. Integrating the inequality (5.4.2) with respect to t for $0\le t\le T$ and change the order of integration in the double integral on the right hand side, we have

$$\begin{aligned}\int_0^t f(s)ds &\le AT+B\int_0^t\int_0^\eta (\eta-s)^{-\alpha}\left(f(s)+\int_0^s f(\tau)d\tau\right)dsd\eta\\ &\le AT+B\int_0^t\left(\int_s^t(\eta-s)^{-\alpha}d\eta\right)\left(f(s)+\int_0^s f(\tau)d\tau\right)ds\\ &\le AT+B\int_0^t\frac{(t-s)^{1-\alpha}}{1-\alpha}\left(f(s)+\int_0^s f(\tau)d\tau\right)ds.\end{aligned}$$

Estimating $(t-s)^{1-\alpha}$ by $T^{1-\alpha}$ for $0<\alpha<1$, we get

$$\int_0^t f(s)ds \;\le\; AT+\frac{BT^{1-\alpha}}{1-\alpha}\int_0^t\left(f(s)+\int_0^s f(\tau)d\tau\right)ds. \tag{5.4.3}$$

Adding the inequalities (5.4.2) and (5.4.3), we have

$$\begin{aligned}f(t)+\int_0^t f(\tau)d\tau &\le A(T+1)+\tilde{B}\int_0^t\left(\frac{1}{(t-s)^\alpha}+1\right)\left(f(s)+\int_0^s f(\tau)d\tau\right)ds\\ &\le A(T+1)+\tilde{B}(T^\alpha+1)\int_0^t\frac{1}{(t-s)^\alpha}\left(f(s)+\int_0^s f(\tau)d\tau\right)ds,\end{aligned}$$

where

$$\tilde{B}=\max\left\{B,\frac{BT^{1-\alpha}}{1-\alpha}\right\}.$$

The required result follows from Lemma 1.3.27.

Propostion 5.4.2 *Let (H1), (H2), (H3') and (H4') hold. If $(x_0, x_1) \in D(A) \times D(A)$, then*

$$\lim_{m\to\infty} \sup_{\{n\geq m,\ 0\leq t\leq T\}} \{\|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha\} = 0.$$

Proof. For $n \geq m$, we have

$$\begin{aligned}
\|f_n(t, u_n, v_n) &- f_m(t, u_m, v_m)\| \\
&\leq \|f_n(t, u_n, v_n) - f_n(t, u_m, v_m)\| + \|f_n(t, u_m, v_m) - f_m(t, u_m, v_m)\| \\
&\leq \|f(t, P^n u_n(t), P^n v_n(t)) - f(t, P^n u_m(t), P^n v_m(t))\| \\
&\quad + \|f(t, P^n u_m(t), P^n v_m(t)) - f(t, P^m u_m(t), P^m v_m(t))\| \\
&\leq F_{\tilde{R}}(T_0)[\|P^n u_n(t) - P^n u_m(t)\|_1 + \|P^n v_n(t) - P^n v_m(t)\|_\alpha \\
&\quad + \|(P^n - P^m)u_m(t)\|_1 + \|(P^n - P^m)v_m(t)\|_\alpha.
\end{aligned}$$

Also,

$$\begin{aligned}
\|(P^n - P^m)v_m(t)\|_\alpha &= \|A^\alpha(P^n - P^m)v_m(t)\| = \|A^{\alpha-\vartheta}(P^n - P^m)A^\vartheta v_m(t)\| \\
&\leq \frac{1}{\lambda_m^{\vartheta-\alpha}}\|(P^n - P^m)A^\vartheta v_m(t)\| \leq \frac{\|A^\vartheta v_m(t)\|}{\lambda_m^{\vartheta-\alpha}}.
\end{aligned}$$

For convenience, we denote

$$\xi_{m,n}(t) = \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha.$$

Thus, we have

$$\begin{aligned}
\|f_n(t, u_n, v_n) &- f_m(t, u_m, v_m)\| \\
&\leq F_{\tilde{R}}(T_0)\left(\xi_{m,n}(t) + \|(P^n - P^m)u_m(t)\|_1 + \frac{\|v_m(t)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}}\right). \qquad (5.4.4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|g_n(t,u_n,v_n) &- g_m(t,u_m,v_m)\| \\ &\leq G_{\tilde{R}}(t)\left(\xi_{m,n}(t)+\|(P^n-P^m)u_m(t)\|_1+\frac{\|v_m(t)\|_\vartheta}{\lambda_m^{\vartheta-\alpha}}\right). \qquad (5.4.5)\end{aligned}$$

From (5.4.5) and Proposition 5.3.3, for $n \geq m$, we have

$$\begin{aligned}&\int_0^s |k(s-\tau)|\|g_n(\tau,u_n,v_n)-g_m(\tau,u_m,v_m)\|d\tau \\ &\quad\leq \|k\|_{L^p(0,T_0)}\|G_{\tilde{R}}\|_{L^q(0,T_0)}\left(\sup_{0\leq\tau\leq s}\|(P^n-P^m)u_m(\tau)\|_1+\frac{V_0}{\lambda_m^{\vartheta-\alpha}}\right) \\ &\quad\quad+\int_0^s |k(s-\tau)|G_{\tilde{R}}(\tau)\xi_{m,n}(\tau)d\tau. \qquad (5.4.6)\end{aligned}$$

Hence, from (5.4.4) and (5.4.6), we get

$$\begin{aligned}&\|f_n(s,u_n,v_n)-f_m(s,u_m,v_m)\| \\ &\quad+\int_0^s |k(s-\tau)|\|g_n(\tau,u_n,v_n)-g_m(\tau,u_m,v_m)\|d\tau \\ &\leq L(R)\left(\sup_{0\leq\tau\leq s}\|(P^n-P^m)u_m(\tau)\|_1+\frac{V_0}{\lambda_m^{\vartheta-\alpha}}\right) \\ &\quad+\left(F_{\tilde{R}}(T_0)\xi_{m,n}(s)+\int_0^s |k(s-\tau)|G_{\tilde{R}}(\tau)\xi_{m,n}(\tau)d\tau\right) \\ &\leq L(R)\left(\sup_{0\leq\tau\leq s}\|(P^n-P^m)u_m(\tau)\|_1+\frac{V_0}{\lambda_m^{\vartheta-\alpha}}\right) \\ &\quad+N\left(\xi_{m,n}(s)+\int_0^s \xi_{m,n}(\tau)d\tau\right), \qquad (5.4.7)\end{aligned}$$

where $N = \max\{F_{\tilde{R}}(T_0), B_k B_{G_{\tilde{R}}}\}$ and B_k and $B_{G_{\tilde{R}}}$ be bounds for k and $G_{\tilde{R}}$, respectively as thesere are bounded functions of t.

Now, from the pair of integral equations (5.3.3), we have

$$\begin{aligned}
\xi_{m,n}(t) \;\leq\; & \int_0^t \|e^{-(t-s)A} - I\| [\|f_n(s,u_n,v_n) - f_m(s,u_m,v_m)\| \\
& \quad + \int_0^s |k(s-\tau)| \|g_n(\tau,u_n,v_n) - g_m(\tau,u_m,v_m)\| d\tau] ds \\
& + \int_0^t \|e^{-(t-s)A}A^\alpha\| [\|f_n(s,u_n,v_n) - f_m(s,u_m,v_m)\| \\
& \quad + \int_0^s |k(s-\tau)| \|g_n(\tau,u_n,v_n) - g_m(\tau,u_m,v_m)\| d\tau] ds.
\end{aligned}$$

Using the estimate (5.4.7), we get

$$\begin{aligned}
\xi_{m,n}(t) \;\leq\; & L(R) \int_0^t \left((M+1) + \frac{C_\alpha}{(t-s)^\alpha} \right) \left(\sup_{0\leq\tau\leq s} \|(P^n - P^m)u_m(\tau)\|_1 + \frac{V_0}{\lambda_m^{\vartheta-\alpha}} \right) ds \\
& + N \int_0^t \left((M+1) + \frac{C_\alpha}{(t-s)^\alpha} \right) \left(\xi_{m,n}(s) + \int_0^s \xi_{m,n}(\tau) d\tau \right) ds \\
\leq\; & C(R,T)B_{mn} + N_1 \int_0^t \frac{1}{(t-s)^\alpha} \left(\xi_{m,n}(s) + \int_0^s \xi_{m,n}(\tau) d\tau \right) ds,
\end{aligned}$$

where

$$B_{mn} = \sup_{0\leq\tau\leq T} \|(P^n - P^m)u_m(\tau)\|_1 + \frac{V_0}{\lambda_m^{\vartheta-\alpha}},$$

$$C(R,T) = L(R) \left((M+1)T + \frac{C_\alpha T^{1-\alpha}}{1-\alpha} \right),$$

$$N_1 = N(T^\alpha + 1)\max\{(M+1), C_\alpha\}.$$

Hence, from Lemma 5.4.1 there exists a constant K such that

$$\xi_{m,n}(t) + \int_0^t \xi_{m,n}(\tau) d\tau \;\leq\; KC(R,T)B_{mn},$$

which implies that

$$\xi_{m,n}(t) \;\leq\; KC(R,T)B_{mn}.$$

Taking supremum over $[0,T]$ and the limit as $m \to \infty$ on both sides, we get the required result since $B_{mn} \to 0$ as $m \to \infty$ provided $\|(P^n - P^m)u_m(t)\|_1 \to 0$ as $m \to \infty$ for $0 \leq t \leq T$.

Now, we prove that for $0 \le t \le T$, $\|(P^n - P^m)u_m(t)\|_1 \to 0$ as $m \to \infty$. We can easily check that for every $x \in H$ and $\eta < 0$,

$$\|A^\eta(P^n - P^m)x\| \le \lambda_m^\eta \|(P^n - P^m)x\| \le \lambda_m^\eta \|x\|. \tag{5.4.8}$$

From the equation (5.4.1), we get

$$\begin{aligned}\|A(P^n - P^m)u_m(t)\| \le\ & \|(P^n - P^m)Ax_0\| + (M+1)\|(P^n - P^m)x_1\| \\ & + (M+1)\int_0^t [\|(P^n - P^m)f_m(s, u_m, v_m)\| \\ & + \int_0^s |k(s-\tau)|\|(P^n - P^m)g_n(\tau, u_n, v_n)\|d\tau]ds.\end{aligned} \tag{5.4.9}$$

We note that in (5.1.3), $f(t,u,v) = f^O(t,u,v) - Bu$ where f^O represents original f appears in (5.1.2). Also, we can check $\|Bu\| \le C\|Au\|$ for $u \in D(A)$ and some constant C. Therefore, we have

$$\begin{aligned}\|(P^n - P^m)f_m(s, u_m, v_m)\| \le\ & \|(P^n - P^m)f_m^O(s, u_m, v_m)\| \\ & + \|A(P^n - P^m)u_m(s)\|.\end{aligned} \tag{5.4.10}$$

Since $A^\beta f_m^O(s, u_m, v_m) \in H$, hence from (5.4.8), we have

$$\begin{aligned}\|(P^n - P^m)f_m^O(s, u_m, v_m)\| &= \|A^{-\beta}(P^n - P^m)A^\beta f_m^O(s, u_m, v_m)\| \\ &\le \frac{1}{\lambda_m^\beta}\|A^\beta f_m^O(s, u_m, v_m)\| \\ &\le \frac{1}{\lambda_m^\beta}\tilde{F}_{\tilde{R}}(T_0).\end{aligned}$$

Similarly, as $A^\beta g_m(s, u_m, v_m) \in H$, we have

$$\|(P^n - P^m)g_m(\tau, u_m, v_m)\| \le \frac{1}{\lambda_m^\beta}\tilde{G}_{\tilde{R}}(\tau).$$

Using these inequalities in (5.4.9), we get

$$\begin{aligned}\|(P^n - P^m)u_m(t)\|_1 &\leq \|(P^n - P^m)Ax_0\| + (M+1)\{\|(P^n - P^m)x_1\| \\ &+ \frac{1}{\lambda_m^\beta} T(\tilde{F}_{\tilde{R}}(T_0) + \|k\|_{L^p(0,T_0)}\|\tilde{G}_{\tilde{R}}\|_{L^q(0,T_0)}) \\ &+ \int_0^t \|(P^n - P^m)u_m(s)\|_1 ds\}.\end{aligned}$$

Gronwall's inequality implies that

$$\begin{aligned}\|(P^n - P^m)u_m(t)\|_1 &\leq [\|(P^n - P^m)Ax_0\| + (M+1)\{\|(P^n - P^m)x_1\| \\ &+ \frac{1}{\lambda_m^\beta} T(\tilde{F}_{\tilde{R}}(T_0)\|k\|_{L^p(0,T_0)}\|\tilde{G}_{\tilde{R}}\|_{L^q(0,T_0)})\}]e^{(M+1)T}\end{aligned}$$

which tend to zero as $m \to \infty$ for $0 \leq t \leq T$. This completes the proof of the proposition.

For the convergence of the solution $(u_n(t), v_n(t))$ of the pair of approximate integral equations (5.4.1), we have the following result.

Theorem 5.4.3 *Let (H1), (H2), (H3') and (H4') hold and let* $(x_0, x_1) \in D(A) \times D(A)$. *Then there exists function* $(u, v) \in X_1(T) \times X_\alpha(T)$ *such that* $(u_n, v_n) \to (u, v)$ *as* $n \to \infty$ *in* $X_1(T) \times X_\alpha(T)$ *and* (u, v) *satisfies (5.3.1) on* $[0, T]$. *Furthermore,* (u, v) *can be extended to the maximal interval of existence* $[0, t_{max})$, $0 < t_{max} \leq \infty$ *satisfying (5.3.1) and* (u, v) *is a unique solution to (5.3.1) on* $[0, t_{max})$.

Proof. Proposition (5.4.2) implies that there exists $(u, v) \in X_1(T) \times X_\alpha(T)$ such that (u_n, v_n) converges to (u, v) in $X_1(T) \times X_\alpha(T)$. Since $(u_n, v_n) \in W_R$ for each n (u, v) is

also in W_R. Further, we have

$$\begin{aligned}
\|f_n(t, u_n, v_n) &- f(t, u(t), v(t))\| \\
&= \|f(t, P^n u_n(t), P^n v_n(t)) - f(t, u(t), v(t))\| \\
&\leq \|f(t, P^n u_n(t), P^n v_n(t)) - f(t, P^n u(t), P^n v(t))\| \\
&\quad + \|f(t, P^n u(t), P^n v(t)) - f(t, u(t), v(t))\| \\
&\leq F_{\tilde{R}}(T_0)[\|P^n u_n(t) - P^n u(t)\|_1 + \|P^n v_n(t) - P^n v(t)\|_\alpha \\
&\quad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha] \\
&\leq F_{\tilde{R}}(T_0)[\|u_n(t) - u(t)\|_1 + \|v_n(t) - v(t)\|_\alpha \\
&\quad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha].
\end{aligned}$$

Taking supremum over $[0, T]$, we get

$$\begin{aligned}
\sup_{0 \leq t \leq T} \|f_n(t, u_n, v_n) &- f(t, u(t), v(t))\| \\
&\leq F_{\tilde{R}}(T_0)[\|u_n - u\|_{X_1(T)} + \|v_n - v\|_{X_\alpha(T)} \\
&\quad + \|(P^n - I)u\|_{X_1(T)} + \|(P^n - I)v\|_{X_\alpha(T)}]. \qquad (5.4.11)
\end{aligned}$$

The right hand side of (5.4.11) tends to zero as n tend to infinity. Similarly

$$\begin{aligned}
\|g_n(t, u_n, v_n) &- g(t, u(t), v(t))\| \\
&= \|g(t, P^n u_n(t), P^n v_n(t)) - g(t, u(t), v(t))\| \\
&\leq \|g(t, P^n u_n(t), P^n v_n(t)) - g(t, P^n u(t), P^n v(t))\| \\
&\quad + \|g(t, P^n u(t), P^n v(t)) - g(t, u(t), v(t))\| \\
&\leq G_{\tilde{R}}(t)[\|P^n u_n(t) - P^n u(t)\|_1 + \|P^n v_n(t) - P^n v(t)\|_\alpha \\
&\quad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha] \\
&\leq G_{\tilde{R}}(t)[\|u_n(t) - u(t)\|_1 + \|v_n(t) - v(t)\|_\alpha \\
&\quad + \|(P^n - I)u(t)\|_1 + \|(P^n - I)v(t)\|_\alpha]. \qquad (5.4.12)
\end{aligned}$$

From (5.4.12), we have

$$\begin{aligned}\sup_{0\le t\le T}\int_0^t |k(t-s)|\|g_n(s,u_n,v_n)-g(s,u(t),v(t))\|ds \\ \le\ L(R)[\|u_n-u\|_{X_1(T)}+\|v_n-v\|_{X_\alpha(T)} \\ +\|(P^n-I)u\|_{X_1(T)}+\|(P^n-I)v\|_{X_\alpha(T)}]\to 0 \text{ as } n\to\infty. \quad (5.4.13)\end{aligned}$$

Hence, using (5.4.11), (5.4.13) and the bounded convergence theorem in (5.4.1), we get

$$\begin{aligned}u(t) &= x_0-(e^{-tA}-I)(A)^{-1}x_1-\int_0^t(e^{-(t-s)A}-I)(A)^{-1}[f(s,u(s),v(s)) \\ &\quad+\int_0^s k(s-\tau)g(\tau,u(\tau),v(\tau))d\tau]ds, \\ v(t) &= e^{-tA}x_1+\int_0^t e^{-(t-s)A}[f(s,u(s),v(s))+\int_0^s k(s-\tau)g(\tau,u(\tau),v(\tau))d\tau]ds.\end{aligned}$$

Thus, for $(x_0,x_1)\in D(A)\times D(A)$, there exists a unique pair of functions $(u,v)\in X_1(T)\times X_\alpha(T)$ such that $(u_n,v_n)\to(u,v)$ as $n\to\infty$ in $X_1(T)\times X_\alpha(T)$ and (u,v) satisfies (5.3.1) on $[0,T]$.

If (u,v) satisfy (5.3.1) on $[0,T]$, then we show that (u,v) can be extended further. Since $0<T_0<\infty$ was arbitrary, we assume that $0<T<T_0$. We consider the equation

$$\begin{aligned}&U''(t)+AU'(t)=F(t,U(t),U'(t))+\int_0^t k(t-s)G(s,U(s),U'(s))ds, \quad t>0, \\ &U(0)=u(T), \quad U'(0)=u'(T),\end{aligned}$$

which can also be written as system of equations

$$\begin{aligned}&U'(t)=V(t), \quad U(0)=u(T), \\ &V'(t)+AV(t)=F(t,U(t),V(t))+\int_0^t k(t-s)G(s,U(s),V(s))ds, \quad V(0)=v(T).\end{aligned}$$

where $F,\ G : [0, T_0 - T] \times X_1 \times X_\alpha \to H$ are defined by

$$\begin{aligned} F(t,x,\tilde{x}) &= f(t+T,x,\tilde{x}) + h(t), \\ G(t,x,\tilde{x}) &= g(t+T,x,\tilde{x}), \end{aligned}$$

where

$$h(t) = \int_0^T k(t+T-s)g(s,u(s),v(s))ds,$$

for $(t,x,\tilde{x}) \in [0,T_0-T] \times X_1 \times X_\alpha$. F and G satisfy (H3) and (H4), respectively, for T_0 replaced by $T_0 - T$. Hence, there exists a $(U,V) \in C([0,T_1], D(A) \times D(A^\alpha))$ for some $0 < T_1 \leq T_0 - T$ satisfying the integral equation

$$\begin{aligned} U(t) &= u(T) + (e^{-tA} - I)(-A)^{-1}v(T) \\ &\quad + \int_0^t (e^{-(t-s)A} - I)(-A)^{-1}[F(s,U(s),V(s)) \\ &\quad + \int_0^s k(s-\tau)G(\tau,U(s),V(s))d\tau]ds, \\ V(t) &= e^{-tA}v(T) + \int_0^t e^{-(t-s)A}[F(s,U(s),V(s)) \\ &\quad + \int_0^s k(s-\tau)G(\tau,U(s),V(s))d\tau]ds, \end{aligned}$$

with $0 \leq t \leq T_1$. Now, we define

$$(\tilde{u}(t),\tilde{v}(t)) = \begin{cases} (u(t),v(t)), & 0 \leq t \leq T, \\ (U(t-T),V(t-T)), & T \leq t \leq T_1 + T. \end{cases}$$

Then, for $t \in [0, T_1 + T]$, $(\tilde{u},\tilde{v})$ satisfies the integral equations

$$\begin{aligned} \tilde{u}(t) &= x_0 + (e^{-tA} - I)(-A)^{-1}x_1 + \int_0^t (e^{-(t-s)A} - I)(-A)^{-1}[f(s,\tilde{u}(s),\tilde{v}(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau]ds, \\ \tilde{v}(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f(s,\tilde{u}(s),\tilde{v}(s)) + \int_0^s k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau]ds. \end{aligned} \tag{5.4.14}$$

To see this, we need to verify (5.4.14) only on $[T, T_1 + T]$. For $t \in [T, T_1 + T]$,

$$\begin{aligned}
\tilde{u}(t) &= U(t-T) \\
&= u(T) + (e^{-(t-T)A} - I)(-A)^{-1}v(T) \\
&\quad + \int_0^{t-T} (e^{-(t-T-s)A} - I)(-A)^{-1}[F(s, U(s), V(s)) \\
&\quad + \int_0^s k(s-\tau)G(\tau, U(\tau), V(\tau))d\tau]ds \\
&= x_0 + (e^{-TA} - I)(-A)^{-1}x_1 + \int_0^T (e^{-(T-s)A} - I)(-A)^{-1}[f(s, u(s), v(s)) \\
&\quad + \int_0^s k(s-\tau)g(\tau, u(\tau), v(\tau))d\tau]ds + (e^{-(t-T)A} - I)(-A)^{-1}(e^{-TA}x_1 \\
&\quad + \int_0^T e^{-(T-s)A}[f(s, u(s), v(s)) + \int_0^s k(s-\tau)g(\tau, u(\tau), v(\tau))d\tau]ds) \\
&\quad + \int_0^{t-T} (e^{-(t-T-s)A} - I)(-A)^{-1}[F(s, U(s), V(s)) \\
&\quad + \int_0^s k(s-\tau)G(\tau, U(\tau), V(\tau))d\tau]ds.
\end{aligned}$$

Putting $s + T = \eta$ and $\tau + T = \xi$, we get

$$\begin{aligned}
\tilde{u}(t) &= x_0 + (e^{-tA} - I)(-A)^{-1}x_1 + \int_0^T (e^{-(t-s)A} - I)(-A)^{-1}[f(s, u(s), v(s)) \\
&\quad + \int_0^s k(s-\tau)g(\tau, u(\tau), v(\tau))d\tau]ds \\
&\quad + \int_T^t e^{-(t-\eta)A} - I)(-A)^{-1}[F(\eta - T, U(\eta - T), V(\eta - T)) \\
&\quad + \int_0^{\eta - T} k(\eta - T - \tau)G(\tau, U(\tau), V(\tau))d\tau]ds \\
&= x_0 + (e^{-tA} - I)(-A)^{-1}x_1 + \int_0^T (e^{-(t-s)A} - I)(-A)^{-1}[f(s, u(s), v(s)) \\
&\quad + \int_0^s k(s-\tau)g(\tau, u(\tau), v(\tau))d\tau]ds \\
&\quad + \int_T^t e^{-(t-\eta)A} - I)(-A)^{-1}[f(\eta, U(\eta - T), V(\eta - T)) + h(\eta - T) \\
&\quad + \int_T^\eta k(\eta - \xi)G(\xi - T, U(\xi - T), V(\xi - T))d\xi]d\eta.
\end{aligned}$$

Since

$$F(s-T,U(s-T),V(s-T)) = f(s,U(s-T),V(s-T)) + h(s-T)$$

and

$$G(s-T,U(s-T),V(s-T)) = g(s,U(s-T),V(s-T)),$$

we have

$$\begin{aligned}
\tilde{u}(t) &= x_0 + (e^{-tA}-I)(-A)^{-1}x_1 + \int_0^t (e^{-(t-s)A}-I)(-A)^{-1}f(s,\tilde{u}(s),\tilde{v}(s))ds \\
&\quad + \int_0^T \int_0^s (e^{-(t-s)A}-I)(-A)^{-1}k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau ds \\
&\quad + \int_T^t \int_0^T (e^{-(t-s)A}-I)(-A)^{-1}k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau ds \\
&\quad + \int_T^t \int_T^s (e^{-(t-s)A}-I)(-A)^{-1}k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau ds \\
&= x_0 + (e^{-tA}-I)(-A)^{-1}x_1 + \int_0^t (e^{-(t-s)A}-I)(-A)^{-1}[f(s,\tilde{u}(s),\tilde{v}(s)) \\
&\quad + \int_0^s k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau]ds.
\end{aligned}$$

Now, for $\tilde{v}$, we have

$$\begin{aligned}
\tilde{v}(t) &= V(t-T) \\
&= e^{-(t-T)A}v(T) + \int_0^{t-T} e^{-(t-T-s)A}[F(s,U(s),V(s)) \\
&\quad + \int_0^s k(s-\tau)G(\tau,U(\tau),V(\tau))d\tau]ds \\
&= e^{-(t-T)A}(e^{-TA}x_1 + \int_0^T e^{-(T-s)A}[f(s,u(s),v(s)) \\
&\qquad + \int_0^s k(s-\tau)g(\tau,u(\tau),v(\tau))d\tau]ds) \\
&\quad + \int_0^{t-T} e^{-(t-T-s)A}[F(s,U(s),V(s)) + \int_0^s k(s-\tau)G(\tau,U(\tau),V(\tau))d\tau]ds.
\end{aligned}$$

Again, putting $T+s=\eta$ and $T+\tau=\xi$ and using the definitions of F and G, we get

$$\begin{aligned}
\tilde{v}(t) &= e^{-tA}x_1 + \int_0^T e^{-(t-s)A}[f(s,u(s),v(s)) + \int_0^s k(s-\tau)g(\tau,u(\tau),v(\tau))d\tau]ds \\
&\quad + \int_T^t e^{-(t-\eta)A}[F(\eta-T,U(\eta-T),V(\eta-T)) \\
&\qquad\qquad + \int_0^{\eta-T} k(\eta-T-\tau)G(\tau,U(\tau),V(\tau))d\tau]ds \\
&= e^{-tA}x_1 + \int_0^T e^{-(t-s)A}[f(s,u(s),v(s)) + \int_0^s k(s-\tau)g(\tau,u(\tau),v(\tau))d\tau]ds \\
&\quad + \int_T^t e^{-(t-\eta)A}[f(\eta,U(\eta-T),V(\eta-T)) + h(\eta-T) \\
&\qquad\qquad + \int_T^{\eta} k(\eta-\xi)G(\xi-T,U(\xi-T),V(\xi-T))d\xi]d\eta \\
&= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}f(s,\tilde{u}(s),\tilde{v}(s))ds \\
&\quad + \int_0^T\int_0^s e^{-(t-s)A}k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau ds \\
&\quad + \int_T^t\int_0^T e^{-(t-s)A}k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau ds \\
&\quad + \int_T^t\int_T^s e^{-(t-s)A}k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau ds \\
&= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f(s,\tilde{u}(s),\tilde{v}(s)) + \int_0^s k(s-\tau)g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))d\tau]ds.
\end{aligned}$$

Thus, we see $(\tilde{u}(t),\tilde{v}(t))$ satisfy (5.3.1) on $[0,T_1+T]$. Hence, we may extend $(u(t),v(t))$ to the maximal interval $[0,t_{max})$ satisfying (5.3.1)on $[0,t_{max})$ with $0<t_{max}\le\infty$.

Now, we show the uniqueness. Let (u_1,v_1) and (u_2,v_2) be two solutions to (5.3.1) on some interval $[0,T]$, where T be any number such that $0<T<t_{max}$.

For convenience, we denote

$$\xi(t) = \|u_1(t)-u_2(t)\|_1 + \|v_1(t)-v_2(t)\|_\alpha.$$

Then, for any $0 < t \leq T$, we have

$$\begin{aligned}\xi(t) \leq & \int_0^t \|e^{-(t-s)A} - I\| [\|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\| \\ & \qquad + \int_0^s |k(s-\tau)| \|g(\tau,u_1(\tau),v_1(\tau)) - g(\tau,u_2(\tau),v_2(\tau))\| d\tau] ds \\ & + \int_0^t \|e^{-(t-s)A}A^\alpha\| [\|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\| \\ & \qquad + \int_0^s |k(s-\tau)| \|g(\tau,u_1(\tau),v_1(\tau)) - g(\tau,u_2(\tau),v_2(\tau))\| d\tau] ds.\end{aligned} \tag{5.4.15}$$

Since

$$\begin{aligned}\|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\| &\leq F_{\tilde{R}}(t_{max})\xi(s), \\ \|g(s,u_1(s),v_1(s)) - g(s,u_2(s),v_2(s))\| &\leq G_{\tilde{R}}(s)\xi(s),\end{aligned}$$

we have

$$\begin{aligned}& \|f(s,u_1(s),v_1(s)) - f(s,u_2(s),v_2(s))\| \\ & \qquad + \int_0^s k(s-\tau)\|g(\tau,u_1(\tau),v_1(\tau)) - g(\tau,u_2(\tau),v_2(\tau))\| d\tau \\ & \qquad \leq F_{\tilde{R}}(t_{max})\xi(s) + \int_0^s |k(s-\tau)| G_{\tilde{R}}(\tau)\xi(\tau) d\tau \\ & \qquad \leq N' \left(\xi(s) + \int_0^s \xi(\tau) d\tau\right),\end{aligned} \tag{5.4.16}$$

where

$$N' = \max\{F_{\tilde{R}}(t_{max}), B_k B_{G_{\tilde{R}}}\}.$$

Using (5.4.16) in (5.4.15), we get

$$\begin{aligned}\xi(t) &\leq N' \int_0^t \left((M+1) + \frac{C_\alpha}{(t-s)^\alpha}\right)\left(\xi(s) + \int_0^s \xi(\tau)d\tau\right) ds \\ &\leq N_2 \int_0^t \frac{1}{(t-s)^\alpha}\left(\xi(s) + \int_0^s \xi(\tau)d\tau\right) ds,\end{aligned}$$

where

$$N_2 = N'(T^\alpha + 1)\max\{(M+1), C_\alpha\}.$$

Hence, from Lemma 5.4.1, we get

$$\xi(t) + \int_0^t \xi(\tau)d\tau = 0,$$

which implies that

$$\xi(t) = 0 \quad \text{for} \quad 0 \le t \le T.$$

From the facts that

$$\begin{aligned} \|u_1(t) - u_2(t)\| &\le \frac{1}{\lambda_0}\|u_1(t) - u_2(t)\|_1, \\ \|u_1(t) - u_2(t)\| &\le \frac{1}{\lambda_0^\alpha}\|u_1(t) - u_2(t)\|_\alpha, \end{aligned}$$

it follows that $(u_1, v_1) = (u_2, v_2)$ on $[0, T]$. Since $0 < T < t_{max}$ was arbitrary, we have $(u_1, v_1) = (u_2, v_2)$ on $[0, t_{max})$. This completes the proof of the theorem.

5.5 Faedo-Galerkin Approximations

For any $0 < T < t_{max}$, we have a unique pair $(u, v) \in X_1(T) \times X_\alpha(T)$ satisfying the integral equations

$$\begin{aligned} u(t) &= x_0 - (e^{-tA} - I)(A)^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)(A)^{-1}[f(s, u(s), v(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau, P^n u_n(\tau), P^n(\tau))d\tau]ds, \\ v(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f(s, u(s), v(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau, P^n u_n(\tau), P^n(\tau))d\tau]ds. \end{aligned} \tag{5.5.1}$$

Also, we have a unique solution $(u_n, v_n) \in X_1(T) \times X_\alpha(T)$ of the pair of approximate integral equations

$$\begin{aligned} u_n(t) &= x_0 - (e^{-tA} - I)A^{-1}x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}[f(s, P^n u_n(s), P^n v_n(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau, P^n u_n(\tau), P^n v_n(\tau))d\tau]ds, \\ v_n(t) &= e^{-tA}x_1 + \int_0^t e^{-(t-s)A}[f(s, P^n u_n(s), P^n v_n(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau, P^n u_n(\tau), P^n v_n(\tau))d\tau]ds. \end{aligned} \tag{5.5.2}$$

If we project the equations (5.5.2) onto H_n, we get the Faedo-Galerkin approximations $(\hat{u}_n(t), \hat{v}_n(t)) = (P^n u_n(t), P^n v_n(t))$ satisfying

$$\begin{aligned} \hat{u}_n(t) &= P^n x_0 - (e^{-tA} - I)A^{-1}P^n x_1 - \int_0^t (e^{-(t-s)A} - I)A^{-1}P^n[f(s, \hat{u}_n(s), \hat{v}_n(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau, \hat{u}_n(\tau), \hat{v}_n(\tau))d\tau]ds, \\ \hat{v}_n(t) &= e^{-tA}P^n x_1 + \int_0^t e^{-(t-s)A}P^n[f(s, \hat{u}_n(s), \hat{v}_n(s)) \\ &\quad + \int_0^s k(s-\tau)g(\tau, \hat{u}_n(\tau), \hat{v}_n(\tau))d\tau]ds. \end{aligned} \tag{5.5.3}$$

The solution (u, v) of (5.5.1) and $(\hat{u}_n, \hat{v}_n)$ of (5.5.3), have the representation

$$\begin{aligned} u(t) &= \sum_{i=0}^{\infty} \alpha_i(t)u_i, \quad \alpha_i(t) = (u(t), u_i), \quad i = 0, 1, \ldots; \\ v(t) &= \sum_{i=0}^{\infty} \beta_i(t)u_i, \quad \beta_i(t) = (v(t), u_i), \quad i = 0, 1, \ldots; \end{aligned} \tag{5.5.4}$$

and

$$\hat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) u_i, \quad \alpha_i^n(t) = (\hat{u}_n(t), u_i), \quad i = 0, 1, \ldots, n;$$

$$\hat{v}_n(t) = \sum_{i=0}^{n} \beta_i^n(t) u_i, \quad \beta_i^n(t) = (\hat{v}_n(t), u_i), \quad i = 0, 1, \ldots, n. \tag{5.5.5}$$

Using (5.5.5) in (5.5.3), we obtain the following systems of the first order integro-differential equations,

$$\begin{aligned}
\frac{d\alpha_i^n(t)}{dt} + \lambda_i \alpha_i^n(t) &= \lambda_i \phi_i + \psi_i + \int_0^t [F_i^n(s, \alpha_0^n(s), ..., \alpha_n^n(s), \beta_0^n(s), ..., \beta_n^n(s)) \\
&+ \int_0^s k(s-\tau) G_i^n(\tau, \alpha_0^n(\tau), ..., \alpha_n^n(\tau), \beta_0^n(\tau), ..., \beta_n^n(\tau)) d\tau] ds, \\
\frac{d\beta_i^n(t)}{dt} + \lambda_i \beta_i^n(t) &= F_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t), \beta_0^n(t), ..., \beta_n^n(t)) \\
&+ \int_0^t k(t-s) G_i^n(s, \alpha_0^n(s), ..., \alpha_n^n(s), \beta_0^n(s), ..., \beta_n^n(s)) ds,
\end{aligned} \tag{5.5.6}$$

with the initial conditions

$$\alpha_i^n(0) = \phi_i, \quad \beta_i^n(0) = \psi_i,$$

where

$$F_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t), \beta_0^n(t), ..., \beta_n^n(t)) = \left(f(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i, \sum_{i=0}^{n} \beta_i^n(t) u_i), u_i \right),$$

$$G_i^n(t, \alpha_0^n(t), ..., \alpha_n^n(t), \beta_0^n(t), ..., \beta_n^n(t)) = \left(g(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i, \sum_{i=0}^{n} \beta_i^n(t) u_i), u_i \right)$$

and $\phi_i = (x_0, u_i)$, $\psi_i = (x_1, u_i)$ for $i = 1, 2, ...n$.

The systems (5.5.6) determine the $\alpha_i^n(t)$'s and $\beta_i^n(t)$'s. Now, we shall show the

convergence of (α_i^n, β_i^n) to (α, β). It can be easily checked that

$$\begin{aligned} A[u(t) - \hat{u}(t)] &= A\left[\sum_{i=0}^{\infty}(\alpha_i(t) - \alpha_i^n(t))u_i\right] \\ &= \sum_{i=0}^{\infty}\lambda_i(\alpha_i(t) - \alpha_i^n(t))u_i \end{aligned}$$

and

$$\begin{aligned} A^{\alpha}[v(t) - \hat{v}(t)] &= A^{\alpha}\left[\sum_{i=0}^{\infty}(\beta_i(t) - \beta_i^n(t))u_i\right] \\ &= \sum_{i=0}^{\infty}\lambda_i^{\alpha}(\beta_i(t) - \beta_i^n(t))u_i. \end{aligned}$$

Thus, we have

$$\|A[u(t) - \hat{u}(t)]\|^2 \geq \sum_{i=0}^{n}\lambda_i^2(\alpha_i(t) - \alpha_i^n(t))^2$$

and

$$\|A^{\alpha}[v(t) - \hat{v}(t)]\|^2 \geq \sum_{i=0}^{n}\lambda_i^{2\alpha}(\beta_i(t) - \beta_i^n(t))^2.$$

We have the following convergence Theorem.

Theorem 5.5.1 *Let (H1), (H2), (H3') and (H4') hold. If $(x_0, x_1) \in D(A) \times D(A)$, then*

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\left\{\sum_{i=0}^{n}\lambda_i^2(\alpha_i(t) - \alpha_i^n(t))^2 + \sum_{i=0}^{n}\lambda_i^{2\alpha}(\beta_i(t) - \beta_i^n(t))^2\right\} = 0.$$

The assertion of Theorem 5.5.1 follows from the facts mentioned above and from the following proposition.

Propostion 5.5.2 *Let (H1), (H2), (H3') and (H4') hold and let T be any number such that $0 < T < t_{max}$, then If $(x_0, x_1) \in D(A) \times D(A)$, then*

$$\lim_{m\to\infty}\sup_{\{n\geq m,\ 0\leq t\leq T\}}\{\|A[\hat{u}_n(t) - \hat{u}_m(t)]\| + \|A^{\alpha}[\hat{v}_n(t) - \hat{v}_m(t)]\|\} = 0.$$

Proof. For $n \geq m$, we have

$$
\begin{aligned}
&\|A(\hat{u}_n(t) - \hat{u}_m(t))\| + \|A^\alpha(\hat{v}_n(t) - \hat{v}_m(t))\| \\
&= \|A(P^n u_n(t) - P^m u_m(t))\| + \|A^\alpha(P^n v_n(t) - P^m v_m(t))\| \\
&\leq \|AP^n(u_n(t) - u_m(t))\| + \|A(P^n - P^m)u_m(t)\| \\
&\quad + \|A^\alpha P^n(v_n(t) - v_m(t))\| + \|A^\alpha(P^n - P^m)v_m(t)\| \\
&\leq \|u_n(t) - u_m(t)\|_1 + \|v_n(t) - v_m(t)\|_\alpha \\
&\quad + \|(P^n - P^m)u_m(t)\|_1 + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\beta v_m\|
\end{aligned}
$$

If $(x_0, x_1) \in D(A) \times D(A)$, then the result follows directly from Proposition 5.4.2.

Chapter 6

Quasi-linear Integro-differential Equations

Geometry may sometimes appear to take the lead over analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on the way

— James Joseph Sylvester

6.1 Introduction

In this chapter, we consider the following quasi-linear implicit integro-differential equation in a Banach space X,

$$\frac{du}{dt}(t) + A(t, u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{6.1.1}$$

where $0 < T < \infty$ and G is a nonlinear Volterra integral operator. In the first section, we prove the existence, uniqueness and continuous dependence on the initial data of a strong solution by using the method of semi-discretization in time which is also known

as the Rothe method or the method of lines. In the second section, we first establish the existence of a unique local mild solution and then prove the existence of a local classical solution using the semigroup theory and the contraction mapping theorem under some different appropriate assumptions.

For applications of the theory of analytic semigroups to related quasi-linear evolution equations, we refer to Amann [1], Lunardy [69] and for applications of the fixed point theorems the reader may refer to Kartsatos [48, 49], Kartsatos and Parrott [50] and references cited therein. Kartsatos [49] and Kartsatos and Parrot [50] have established the existence and uniqueness of a solution and used the Galerkin method for the approximation of the solution to the quasi-linear functional problem

$$u'(t) + A(t, u_t)u(t) = 0, \quad t \in [0, T), \quad u(0) = \phi,$$

in a Banach space X, where $A(t, \phi)v$ is m-accretive in $v \in D$ (fixed) for every pair $(t, \phi) \in [0, T) \times C_0$, D is a subset of X, C_0 is a closed subset of the space of all continuous functions ϕ on $(-\infty, 0]$ into D with $\|\phi\|_\infty \leq r$ for some fixed $r > 0$ and $u_t \in C_0$, given by $u_t(s) = u(t + s)$, $s \in (-\infty, 0]$. Murphy [73] has constructed a family of approximate solutions to the homogeneous quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = 0, \quad t \in (0, T], \quad u(0) = \phi. \tag{6.1.2}$$

He showed that the approximate solution converges to a *limit solution* and this *limit solution* becomes a unique solution to this homogeneous case of (6.1.1). Kartsatos [48] established a theorem concerning the existence of a unique strong solution to (6.1.2) under the assumption that $A(t, u)v$ is Lipschitzian in t, u and m-accretive in v. In the second section, the condition on $A(t, u)v$ is motivated by Kartsatos [48].

For the study of a particular case of (6.1.1), in which $f(t, u, v) = f(t, u)$, that is,

the abstract quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = f(t, u(t)), \quad t \in (0, T], \quad u(0) = \phi, \tag{6.1.3}$$

we refer to T. Kato [61, 58, 60, 62], S. Kato [54], Amann [1] and references cited therein. The crucial assumption in these papers is that the linear operator $A(t, u)$, depending on t and the unknown u, has the dual property of being the negative generator of a C_0-semigroup but not necessarily of an analytic semigroup and at the same time, a bounded operator on Y into X. Moreover, it is assumed in [61, 58, 60, 62] that there is an isomorphism S of Y onto X with the property that

$$SA(t, w)S^{-1} = A(t, w) + B(t, w),$$

where $B(t, w)$ is in the space $B(X)$ of all bounded linear operators from X into X. In [54], it is assumed that there is a family $\{S(t, w)\}$ of isomorphisms of Y onto X such that

$$S(t, w)A(t, w)S(t, w)^{-1} = A(t, w) + B(t, w),$$

where $B(t, w) \in B(X)$ and $S(t, w)$ satisfies the Lipschitz-like condition

$$\|S(t, w_1) - S(s, w_2)\|_{Y,X} \leq \mu(|t - s| + \|w_1 - w_2\|_X),$$

where μ is a constant. T. Kato [61] has proved the existence, uniqueness and continuous dependence on the initial data of a solution of an abstract quasi-linear and showed that these results are applicable to different kinds of quasi-linear equations such as symmetric hyperbolic systems of the first order, the wave equation, the Korteweg-de Vries equation, the Navier-Stokes and the Euler equation, the magnetohydrodynamics equation, the coupled Maxwell and Dirac equations, etc. The results in [61] are based on the theory of linear 'hyperbolic' equations which was developed in [57, 59]. S. Kato [54] has proved the existence of a strong solution to (6.1.3) under certain conditions on $A(t, u)$ and $f(t, u)$ for $(t, u) \in J \times W$ similar to that of Crandal and Sougandis [28].

Amann [1] has treated various cases of (6.1.3) in interpolation spaces using the theory of analytic semigroups.

Using the Rothe method, Kačur [47] has proved the existence and uniqueness results for the nonlinear implicit integro-differential equation

$$\frac{du}{dt}(t) + Au(t) = G(t, K(u)(t)), \quad \text{a.e.} \quad t \in (0,T),$$
$$u = \phi \quad \text{on} \quad [-q,T], \quad q > 0,$$

in a Hilbert space H with the assumption that $A : V \to V^*$ is a coercive maximal monotone operator where V is a reflexive Banach space with the dual V^* such that $V \cap H$ is dense in V and H and the function f and the Volterra operator $K(u)(t)$ satisfy certain Lipschitz-like conditions.

The existence, uniqueness and continuous dependence on the initial data of a solution to some abstract nonlinear implicit and explicit integro-differential equations have been studied by Bahuguna and Raghavendra [8, 9] by using the Rothe method under some appropriate conditions. Bahuguna [6] has also proved similar results by using the Rothe method for the abstract quasi-linear explicit integro-differential equation

$$\frac{du}{dt}(t) + A(u(t))u(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t),$$

for $0 < t < T$ with the initial value u_0 in a real reflexive Banach space X, whose dual is uniformly convex, $A(u)$ is a linear operator in X for each u in an open subset W of Y where Y is also a real reflexive Banach space which is continuously and compactly embedded in X, a is a real-valued and f is a Y-valued function defined on $[0,T]$ and k is a Y-valued map defined on $[0,T] \times W$ while using the semigroup theory together with the contraction mapping theorem, Bahuguna [7] has established the existence of a unique classical solution to the quasi-linear integro-differential equation in Banach

space X,

$$\frac{du}{dt}(t) + A(t, u(t))u(t) = \int_0^t a(t-s)k(s, u(s))ds + f(t),$$
$$u(0) = u_0,$$

where $A(t, u)$ is a linear operator in X for each u in an open subset W of Y, maps a, f, k and spaces X, Y have same properties as in [6]. Also using similar techniques of paper [6], in [12], we have established the existence, uniqueness and continuous dependence of a strong solution to the quasi-linear "implicit" integro-differential equation

$$\frac{du}{dt}(t) + A(u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0,$$

in a reflexive Banach space whose dual is uniformly convex.

6.2 Method of Semi-discretization in Time

In this section, we shall prove the existence of a unique strong solution to the quasi-linear integro-differential equation and its continuous dependence on the initial data.

Let X and Y be two real reflexive Banach spaces such that the embedding $Y \hookrightarrow X$ is dense and continuous. Consider the following abstract quasi-linear implicit integro-differential equation

$$\frac{du}{dt}(t) + A(t, u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{6.2.1}$$

in a Banach space X. Here $0 < T < \infty$, $A(t, u)$ is a linear operator in X for each (t, u) in $[0, T] \times W$, where W is an open subset of Y, G is a nonlinear Volterra integral operator defined from $C([0, T]; X)$ into $C([0, T]; X)$ and the nonlinear map f is defined

from $[0, T] \times Y \times Y$ into Y. By a *strong solution* to (6.2.1) on $[0, T']$, $0 < T' \leq T$, we mean an absolutely continuous function u from $[0, T']$ into X such that $u(t) \in W$ for almost every (a.e. in short) $t \in [0, T']$ and satisfies (6.2.1) almost everywhere (a.e. in short) on $[0, T']$.

In this section, we shall use the Rothe method to establish the existence and uniqueness results. The Rothe method, introduced by E. Rothe in 1930, is a very efficient tool for proving the existence and uniqueness of a solution to a linear, nonlinear parabolic or a hyperbolic problem of higher order. It has been further developed by Kačur [47], Nečas [74], Rektorys [81], Kartsatos and Ziglar [53], Bahuguna and Raghavendra [6, 8, 9] and others. This method consists in replacing the time derivatives in an evolution equation by the corresponding difference quotients giving rise to a system of time independent operator equations. With the help of the theory of semigroups and the theory of monotone operators, each of these systems is guaranteed to have a unique solution. An approximate solution to the evolution is defined in terms of the solutions of these time independent systems. After proving *apriori* estimates for the approximate solution, the convergence of the approximate solution to the unique solution of the evolution equation is established. In these works, either global Lipschitz conditions or local Lipschitz conditions with some growth conditions on nonlinear forcing terms have been assumed.

In the present work, we assume only local Lipschitz conditions on the nonlinear maps f and G. We first prove that the discrete points lie in a ball in X of fixed radius R where R is independent of the discretization parameters. Then, using the local Lipschitz continuity, we establish *apriori* estimates on the difference quotients. With the help of these *apriori* estimates, we prove the convergence of a sequence of approximate solutions, defined in terms of the discrete points, to a unique solution of the problem.

Below, we state the preliminaries, the assumptions and the main result Theorem 6.2.1. After that we state and proved the basic lemmas which are used in the proof of the main result. We conclude this section with a proof of Theorem 6.2.1.

Let X and Y be as earlier assumed. Let $\|x\|_Z$ denote the norm of an element x belonging to a Banach space Z. For $r > 0$, let $B_Z(x,r)$ denote the open ball $\{z \in Z : \|z - x\|_Z < r\}$ of the radius r and let $\bar{B}_Z(x,r)$ be its closure. For an interval J of the real numbers, we denote by $C(J;Z)$, $WC(J;Z)$, $Lip(J;Z)$ and $ABS(J;Z)$ the spaces of all continuous, weakly continuous, Lipschitz continuous, and absolutely continuous functions from J into Z, respectively.

For a real β, $N(Z,\beta)$ represents the set of all densely defined linear operators L in Z such that if $\lambda > 0$ and $\lambda\beta < 1$, then $(I + \lambda L)$ is one-to-one with a bounded inverse defined everywhere on Z and

$$\|(I + \lambda L)^{-1}\|_{B(Z)} \;\leq\; (1 + \lambda\beta)^{-1}, \tag{6.2.2}$$

where I is the identity operator on Z. The Hille-Yosida (1.3.11) theorem states that $L \in N(Z,\beta)$ if and only if $-L$ is the infinitesimal generator of a C_0-semigroup e^{-tL}, $t \geq 0$, on Z satisfying $\|e^{-tL}\|_{B(Z)} \leq e^{\beta t}$, $t \geq 0$. A linear operator L on $D(L) \subseteq Z$ into Z is said to be accretive in Z if for every $u \in D(L)$,

$$< Lu, u^* > \;\geq\; 0 \quad \text{for some } u^* \in F(u),$$

where $< z, z^* >$ is the value of $z^* \in Z^*$ at $z \in Z$ and $F : Z \to 2^{Z^*}$ is the duality map given by

$$F(z) = \{z^* \in Z^* \;:\; < z, z^* > \;=\; \|z\|_Z^2 = \|z^*\|_{Z^*}^2\}.$$

Here 2^{Z^*} denotes the power set of Z^*. If $L \in N(Z,\beta)$, then the Lumer-Phillips Theorem 1.3.14 implies that $(L+\beta I)$ is m-accretive in Z, i.e., $(L+\beta I)$ is accretive and the range

$R(L + \lambda I) = Z$ for some $\lambda > \beta$. If Z^* is uniformly convex, then F is single valued and uniformly continuous on bounded subsets of Z.

We assume, in addition, that the embedding $Y \hookrightarrow X$ is compact and the dual X^* is uniformly convex. Further, we make the following hypotheses.

(H1) There exist an open subset W of Y and $\beta \geq 0$ such that $u_0 \in W$ and

$$A : [0, T] \times W \to N(X, \beta).$$

(H2) For each $t, w \in [0.T] \times W$, we have $A(t, w) \in B(Y, X)$ for each $w \in W$, $t \to A(t, w)$ is the Lipschitz continuous with the Lipschitz constant L_A in B(Y,X)-norm.There exist the positive constants μ_A and γ_A such that for all w, w_1, $w_2 \in W$, $t \in [0, T]$ and $v \in Y$, we have $Y \subseteq D(A(t, w))$,

$$\begin{aligned} \|(A(t, w_1) - A(t, w_2))v\|_X &\leq \mu_A \|w_1 - w_2\|_X \|v\|_Y, \\ \|A(t, w)v\|_X &\leq \gamma_A \|v\|_Y. \end{aligned}$$

(H3) There exist a linear isometric isomorphism $S : Y \to X$, a map $P : [0, t] \times W \to B(X)$ and the positive constants μ_P and γ_P such that for all w, w_1, $w_2 \in W$ and $t \in [0, t]$,

$$\begin{aligned} SA(t, w) = A(t, w)S + P(t, w)S, \quad \|P(t, w)\|_X &\leq \gamma_P, \\ \|P(t, w_1) - P(t, w_2)\|_X &\leq \mu_P \|w_1 - w_2\|_Y. \end{aligned}$$

(H4) The nonlinear map $G : C([0, T]; X) \to C(0, T; X)$ satisfies

(a) For all $u, v \in \bar{B}_{C([0,T];X)}(\tilde{u}_0, r)$,

$$\|G(u) - G(v)\|_{C([0,T];X)} \leq \mu_G(r) \|u - v\|_{C([0,T];X)},$$

where $\mu_G(r)$ is a nonnegative, nondecreasing function and $\tilde{u_0} \in C([0,T];X)$ be defined by $\tilde{u_0}(t) = u_0$ for all $t \in [0,T]$.

(b) For all t, $s \in [0,T]$ and $u \in Lip([0,T];X) \cap \bar{B}_{C([0,T];X)}(\tilde{u_0}, r)$,

$$\|G(u)(t) - G(u)(s)\|_X \leq \gamma_G(r)|t-s|(1 + \|du/dt\|_{L^\infty([0,T];X)}),$$

where $\gamma_G(r)$ is a nonnegative and nondecreasing function.

(c) Furthermore, the subspace $C([0,T];Y)$ of space $C([0,T];X)$ is an invariant subspace of the map G, i.e., $G : C([0,T];Y) \to C([0,T];Y)$ which satisfy

$$\|G(u)(t)\|_Y \leq \lambda_G(r) \quad \text{for} \quad u \in B_Y(u_0, r),$$

where $\lambda_G(r)$ is a nonnegative and nondecreasing function.

In particular, we may take the operator G as a Volterra operator defined by

$$G(u)(t) = \int_0^t a(t-s)k(s, u(s))ds$$

in which the function a is a real valued continuous function defined on $[0,T]$ and k is defined on $[0,T] \times Y$ into Y and $\|k(t,w)\|_Y \leq C_k$ for every $(t,w) \in [0,T] \times Y$, then map G satisfies hypotheses (c).

(H5) The nonlinear map $f : [0,T] \times Y \times Y \to Y$ is a bounded function:

$$\|f(t,u,v)\|_Y \leq \lambda_f(r),$$

for all $(t,u,v) \in [0,T] \times Y \times Y$ with $\|u\|_Y + \|v\|_Y \leq r$, where $\lambda_f(r)$ is a nonnegative function. Also, this map satisfies the Lipschitz condition

$$\|f(t_1,u_1,v_1) - f(t_2,u_2,v_2)\|_X \leq \mu_f(r)[|\phi(t_1) - \phi(t_2)| + \|u_1 - u_2\|_X + \|v_1 - v_2\|_X],$$

for all t_1, $t_2 \in [0,T]$ and all u_i, $v_i \in \bar{B}_X(u_0,r)$, $i = 1,2$, where ϕ is a real-valued continuous function of the bounded variation on $[0,T]$ and $\mu_f(r)$ is a nonnegative and nondecreasing function.

Let $R > 0$ be such that $W_R = \bar{B}_Y(u_0,R) \subseteq W$. We set

$$\begin{aligned} R_0 &= \frac{R}{3}(1+e^{2\theta T})^{-1}, && (6.2.3)\\ R_1 &= \max\{R, \lambda_G(3R)+\|u_0\|_Y\}, && (6.2.4)\\ M &= \lambda_f(R+\|u_0\|+\lambda_G(3R)), && (6.2.5)\end{aligned}$$

where $\theta = \beta + \|P\|_{B(X)}$ and $V(\phi)$ is the total variation of ϕ on $[0,T]$.

Let $z_0 \in Y$ and T_0, $0 < T_0 \leq T$ be such that

$$\begin{aligned} \|Su_0 - z_0\|_X &\leq R_0, && (6.2.6)\\ T_0[\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M] &\leq R_0. && (6.2.7)\end{aligned}$$

We note that (6.2.6) and (6.2.7) imply that

$$(1+e^{2\theta T})[\|Su_0 - z_0\| + T_0\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M\}] \leq 2R/3. \qquad (6.2.8)$$

We have the following main result for the existence, uniqueness and continuous dependence on the initial data of a strong solution to (6.2.1).

Theorem 6.2.1 *Let (H1)-(H5) hold. Then there exists a unique strong solution u to (6.2.1) on $[0,T_0]$ such that $u \in Lip([0,T_0];X)$. Furthermore, if $v_0 \in \bar{B}_Y(u_0,R_0)$, then there exists a strong solution v to (6.2.1) on $[0,T_0]$ with the initial point u_0 replaced by v_0 and*

$$\|u(t)-v(t)\|_X \leq C\|u_0 - v_0\|_X, \quad t \in [0,T_0],$$

where C is a positive constant depending only on T_0.

To apply the Rothe method, we proceed in the following way. Let $w_0 = Su_0$. Let $h = T_0/n$ for all the positive integers $n \geq N$, where N is a positive integer such that $\theta(T_0/N) < 1/2$. For $n \geq N$, we set $u_0^n = u_0$, $\tilde{u}_0^n = \tilde{u}_0$ and $t_j^n = jh$ for $j = 1, 2, \ldots, n$. We consider the scheme

$$\delta u_j^n + A(t_j^n, u_{j-1}^n)u_j^n = f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)), \quad j = 1, 2, \ldots, n, \tag{6.2.9}$$

where, for $j = 1, 2, \ldots, n$, $n \geq N$,

$$\delta u_j^n = (u_j^n - u_{j-1}^n)/h, \tag{6.2.10}$$

$$\tilde{u}_j^n(t) = \begin{cases} u_0, & t = 0, \\ u_{i-1}^n + (1/h)(t - t_{i-1}^n)(u_i^n - u_{i-1}^n), & t \in [t_{i-1}^n, t_i^n], \\ & \quad i = 1, 2, \ldots, j, \\ u_j^n, & t \in [t_j^n, T_0]. \end{cases} \tag{6.2.11}$$

The following result establishes the fact that $u_j^n \in W_R$, $j = 1, 2, \ldots, n$, $n \geq N$.

Lemma 6.2.2 *For each $n \geq N$, there exist a unique $u_j^n \in W_R$, $j = 1, 2, \ldots, n$ satisfying (6.2.9).*

Proof. It follows from Lemma 1.3.28 that there exists a unique $u_1^n \in Y$ such that

$$u_1^n + hA(t_1^n, u_0)u_1^n = u_0 + hf(t_1^n, u_0, G(\tilde{u}_0)(t_1^n)). \tag{6.2.12}$$

Applying S on both the sides in (6.2.12) using (H3) and putting $w_1^n = Su_1^n$, we have

$$\begin{aligned}(w_1^n - z_0) + hA(t_1^n, u_0)(w_1^n - z_0) &+ hP(t_1^n, u_0)(w_1^n - z_0) \\ = (w_0 - z_0) - hA(t_1^n, u_0)z_0 &+ hP(t_1^n, u_0)z_0 + hSf(t_1^n, u_0, G(\tilde{u}_0)(t_1^n)).\end{aligned}$$

The estimates in Lemma 1.3.28 imply that

$$\|w_1^n - z_0\|_X \;\leq\; (1-h\theta)^{-1}[\|w_0 - z_0\|_X + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Since $h\theta < 1/2$, we have

$$\|w_1^n - z_0\|_X \;\leq\; e^{2\theta h}[\|w_0 - z_0\|_X + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Therefore

$$\|w_1^n - w_0\|_X \;\leq\; (1+e^{2\theta h})[\|w_0 - z_0\| + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)] \;\leq\; R,$$

in view of the estimate (6.2.8). Hence $u_1^n \in W_R$. Now, suppose that $u_i^n \in W_R$ for $i = 1, 2, \ldots, j-1$. Again, Lemma 1.3.28 implies that for $2 \leq j \leq n$, there exist a unique $u_j^n \in Y$ such that

$$u_j^n + hA(t_j^n, u_{j-1}^n)u_j^n = u_{j-1}^n + hf(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)). \tag{6.2.13}$$

Proceeding as before and putting $w_j^n = Su_j^n$, we get the estimate

$$\|w_j^n - z_0\|_X \;\leq\; e^{2\theta h}[\|w_{j-1} - z_0\|_X + h(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Reiterating the above inequality, we get

$$\|w_j^n - z_0\|_X \;\leq\; e^{2\theta jh}[\|w_0 - z_0\|_X + jh(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)].$$

Using the fact that $jh \leq T_0$, we arrive at

$$\|w_j^n - w_0\|_X \;\leq\; (1+e^{2\theta T})[\|w_0 - z_0\|_X + T_0(\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M)] \;\leq\; R.$$

The above estimate and the equation (6.2.12) and (6.2.13) gives the required result. This completes the proof of the lemma.

Lemma 6.2.3 *There exists a positive constant C, independent of j, h and n such that*

$$\|\delta u_j\|_X \leq C, \quad j = 1, 2, \ldots, n; \quad n \geq N.$$

Proof. Putting $j = 1$ in (6.2.9), we get

$$\delta u_1^n + hA(t_1^n, u_0)(\delta u_1^n) = -A(t_1^n, u_0)u_0 - f(t_1^n, u_0, G(\tilde{u}_0)(t_1^n)).$$

Using Lemma 1.3.28, we have

$$\|\delta u_1^n\|_X \leq e^{2\theta T}[\gamma_A \|u_0\|_Y + M] := C_0. \tag{6.2.14}$$

From (6.2.9) for $2 \leq j \leq n$, we have

$$\begin{aligned} \delta u_j^n &+ hA(t_j^n, u_{j-1}^n)(\delta u_j^n) \\ &= \delta u_{j-1}^n - (A(t_j^n, u_{j-1}^n) - A(t_{j-1}^n, u_{j-2}^n))u_{j-1}^n \\ &\quad + f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n)). \end{aligned}$$

Using Lemma 1.3.28 again, we have

$$\begin{aligned} \|\delta u_j^n\|_X \leq\ & e^{2h\theta}[\|\delta u_{j-1}^n\|_X + \|(A(t_j^n, u_{j-1}^n) - A(t_{j-1}^n, u_{j-2}^n))u_{j-1}^n\|_X \\ & + \|f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n))\|_X]. \end{aligned}$$

Adding and subtracting $A(t_{j-1}^n, u_{j-2}^n)u_{j-1}^n$, we have

$$\begin{aligned} \|\delta u_j^n\|_X \leq\ & e^{2h\theta}[\|\delta u_{j-1}^n\|_X + \|A(t_j^n, u_{j-1}^n) - A(t_{j-1}^n, u_{j-1}^n)\|_X \|u_{j-1}^n\|_X \\ & + \|(A(t_{j-1}^n, u_{j-1}^n) - A(t_{j-1}^n, u_{j-2}^n))u_{j-1}^n\|_X \\ & + \|f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n))\|_X]. \end{aligned}$$

Now, using (H2), we get

$$\begin{aligned}\|\delta u_j^n\|_X &\leq e^{2h\theta}[\|\delta u_{j-1}^n\|_X + hL_A\|u_{j-1}^n\|_X + \mu_A h\|\delta u_{j-1}^n\|_X\|u_{j-1}^n\|_Y \\ &\quad + \|f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n))\|_X].\end{aligned}$$

We have

$$\begin{aligned}\|\delta u_j^n\|_X &\leq e^{2\theta h}\left[(1+C_1h)\|\delta u_{j-1}^n\|_X + C_2h\right. \\ &\quad + \left.\|f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n))\|_X\right],\end{aligned} \tag{6.2.15}$$

for some positive constants $C_1$and C_2 independent of j, h and n. We note that

$$\begin{aligned}&\|G(\tilde{u}_{j-1}^n)(t_j^n) - G(\tilde{u}_{j-2}^n)(t_{j-1}^n)\|_X \\ &\leq \mu_G(3R)h\|\delta u_{j-1}^n\|_X + \gamma_G(3R)h(1 + \max_{1\leq i\leq j-1}\|\delta u_i^n\|_X),\end{aligned} \tag{6.2.16}$$

$$\begin{aligned}&\|f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) - f(t_{j-1}^n, u_{j-2}^n, G(\tilde{u}_{j-2}^n)(t_{j-1}^n))\|_X \\ &\leq \mu_f(R_1)[|\phi(t_j^n) - \phi(t_{j-1}^n)| + h\|\delta u_{j-1}^n\|_X + \mu_G(3R)h\|\delta u_{j-1}^n\|_X \\ &\quad + \gamma_G(3R)h(1 + \max_{1\leq i\leq j-1}\|\delta u_i^n\|_X)].\end{aligned} \tag{6.2.17}$$

Using (6.2.16) and (6.2.17) in (6.2.15), we obtain

$$\begin{aligned}\|\delta u_j^n\|_X &\leq e^{2h\theta}[(1+hC_1)\|\delta u_{j-1}^n\|_X + hC_2 + \mu_f(R_1)\{|\phi(t_j^n) - \phi(t_{j-1}^n)| \\ &\quad + h\|\delta u_{j-1}^n\|_X\mu_G(3R)h\|\delta u_{j-1}^n\|_X + \gamma_G(3R)h(1 + \max_{1\leq i\leq j-1}\|\delta u_i^n\|_X)\}] \\ &\leq e^{2h\theta}\left[(1+C_3h)\max_{1\leq i\leq j-1}\|\delta u_i^n\|_X + C_4h + C_4|\phi(t_j^n) - \phi(t_{j-1}^n)|\right].\end{aligned}$$

Hence

$$\max_{1\leq i\leq j} \|\delta u_i^n\|_X \leq e^{2\theta h}(1+C_5h)\left[\max_{1\leq i\leq j-1} \|\delta u_i^n\|_X + C_5|\phi(t_j^n) - \phi(t_{j-1}^n)| + C_5h\right], \tag{6.2.18}$$

where C_5 is another positive constant independent of j, h and n. Reiterating the inequality (6.2.18), we get

$$\max_{1\leq i\leq j} \|\delta u_i^n\|_X \leq e^{2\theta jh}(1+C_5h)^j[\|\delta u_1^n\|_X + C_5V(\phi) + C_5T_0],$$

where $V(\phi)$ is the total variation of ϕ. Using (6.2.14), we have

$$\|\delta u_i^n\|_X \leq e^{2(\theta+C_5)T}[C_0 + C_5V(\phi) + C_5T_0] := C.$$

This completes the proof of the Lemma.

Now, we define the Rothe sequence of the functions $\{U^n\}$ from J_0 into Y by

$$U^n(t) = u_{j-1}^n + \frac{(t-t_{j-1})}{h}(u_j^n - u_{j-1}^n), \quad t \in [t_{j-1}^n, t_j^n], \quad j = 1, 2, \ldots, n. \tag{6.2.19}$$

Furthermore, we define a sequence of the step functions $\{X^n\}$ from $(-h, T_0]$ into Y given by

$$X^n(t) = \begin{cases} u_0, & t \in (-h, 0], \\ u_j, & t \in (t_{j-1}^n, t_j^n], \quad j = 1, 2, \ldots, n. \end{cases} \tag{6.2.20}$$

Remark 6.2.4 We observe that $X^n(t) \in W_R$ for all $t \in (-h, T_0]$ and $n \geq N$. Also, $X^n(t) - U^n(t) \to 0$ in X uniformly on J_0 as $n \to \infty$ and $\{U^n\}$ are in $Lip(J_0, X)$ with the uniform Lipschitz constant C.

Also, define

$$f^n(t) \;=\; f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)), \quad t \in (t_{j-1}^n, t_j^n), \quad j = 1, 2, \ldots, n. \tag{6.2.21}$$

and

$$A^n(t, u) \;=\; A(t_j^n, u), \quad t \in (t_{j-1}^n, t_j^n), \quad j = 1, 2, \ldots, n. \tag{6.2.22}$$

Lemma 6.2.5

(a) $$\int_0^t A^n(s, X^n(s-h))X^n(s)ds \;=\; u_0 - U^n(t) + \int_0^t f^n(s)ds. \tag{6.2.23}$$

(b) $$\frac{d^-}{dt} U^n(t) + A^n(t, X^n(t-h))X^n(t) = f^n(t) \quad \textit{for} \quad t \in (0, T]. \tag{6.2.24}$$

Proof. *(a)* Using the definitions (6.2.19), (6.2.20), (6.2.21), (6.2.22) and the equation (6.2.9), we have

$$\begin{aligned}
&\int_0^t A^n(s, X^n(s-h))X^n(s)ds \\
&= \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} A^n(s, X^n(s-h))X^n(s)ds + \int_{t_{j-1}}^{t} A^n(s, X^n(s-h))X^n(s)ds \\
&= \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} A(t_i, u_{i-1})u_i ds + \int_{t_{j-1}}^{t} A(t_j, u_{j-1})u_j ds \\
&= \sum_{i=1}^{j-1} \int_{t_{i-1}}^{t_i} \left(-\frac{(u_i - u_{i-1})}{h} + F(t_i^n, u_{i-1}^n, G(\tilde{u}_{i-1}^n)(t_i^n)) \right) ds \\
&\quad + \int_{t_{j-1}}^{t} \left(-\frac{(u_j - u_{j-1})}{h} + F(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n)) \right) ds \\
&= -\sum_{i=1}^{j-1} (u_i - u_{i-1}) - \frac{t - t_{j-1}}{h}(u_j - u_{j-1}) + \int_0^t F^n(s)ds \\
&= u_0 - U^n(t) + \int_0^t F^n(s)ds.
\end{aligned}$$

(b) For $t \in (t_{j-1}, t_j)$,

$$A^n(t, X^n(t-h))X^n(t) = A(t_j^n, X^n(t-h))X^n(t) = A(t_j^n, u_{j-1})u_j$$

and

$$\frac{d^-}{dt}U^n(t) = \frac{(u_j - u_{j-1})}{h}.$$

Therefore

$$\begin{aligned} \frac{d^-}{dt} U^n(t) + A^n(t, X^n(t-h))X^n(t) &= \frac{(u_j - u_{j-1})}{h} + A(t_j^n, u_{j-1})u_j \\ &= f(t_j^n, u_{j-1}^n, G(\tilde{u}_{j-1}^n)(t_j^n) = f^n(t). \end{aligned}$$

The proof of the lemma is complete.

Lemma 6.2.6 *There exist a subsequence $\{U^m\}$ of $\{U^n\}$ and a function u in $Lip(J_0, X)$ such that $U^m \to u$ in $C(J_0, X)$ (with the supremum norm) as $m \to \infty$.*

Proof. Since, $\{X^n(t)\}$ is uniformly bounded in Y, the compact embedding of Y, implies there exist a subsequence $\{X^m\}$ of $\{X^n\}$ and a function $u : J_0 \to X$ such that $X^m(t) \to u(t)$ in X as $m \to \infty$. The reflexivity of Y implies that $u(t)$ is the weak limit of X^m in Y, hence $u(t)$ lies in Y (in fact in W_R since $X^m(t)$ is in W_R). Now, $X^m(t) - U^m(t) \to 0$ in X, so $U^m(t) \to u(t)$ as $m \to \infty$. The uniform continuity of $\{U^n\}$ on J_0 implies that $\{X^m\}$ is an equicontinuous family in $C(J_0, X)$ and the strong convergence of $U^m(t)$ to $u(t)$ in X implies that $\{U^m\}$ is relatively compact in X. We use the Ascoli-Arzela theorem to conclude that $U^m \to u$ in $C(J_0, X)$ as $m \to \infty$. Since U^m are in $Lip(J_0, X)$ with a uniform Lipschitz constant, $u \in Lip(J_0, X)$. This completes the proof of the lemma.

Proof of the Theorem 6.2.1. First, we show that $A^m(t, X^m(t-h))X^m(t) \rightharpoonup A(u(t))u(t)$ in X as $m \to \infty$, where '$\rightharpoonup$' denotes the weak convergence in X.

$$\begin{aligned}
&A^m(t, X^m(t-h))X^m(t) - A(t, u(t))u(t) \\
&= (A^m(t, X^m(t-h)) - A(t, u(t)))X^m(t) + A(t, u(t))(X^m(t) - u(t)).
\end{aligned}$$

Now, for $t \in (t^m_{j-1}, t^m_j]$, $j = 1, 2, \cdots, m$;

$$\begin{aligned}
&\|(A^m(t, X^m(t-h)) - A(t, u(t)))X^m(t)\|_X \\
&= \|(A(t^m_j, u^m_{j-1}) - A(t, u(t)))u^m_j\|_X \\
&\leq \|(A(t^m_j, u^m_{j-1}) - A(t^m_j, u(t)))u^m_j\|_X + \|(A(t^m_j, u(t)) - A(t, u(t)))u^m_j\|_X \\
&\leq \mu_A(R + \|u_0\|_Y)\|u^m_{j-1} - u(t)\|_X + L_A|t^m_j - t|\|u^m_j\|_X \\
&= \mu_A(R + \|u_0\|_Y)\|X^m(t-h) - u(t)\|_X + L_A|t^m_j - t|\|X^m(t)\|_X \to 0,
\end{aligned}$$

as $m \to \infty$, since $X^m(t) \to u(t)$ in X uniformly on J_0 and $t^m_j \to t$. Since $A(t, u(t)) \in N(X, \beta)$, $\beta I + A(t, u)$ is m-accretive in X. We use Lemma 1.3.31 and the fact that

$$\|A(t, u(t))(X^m(t) - u(t))\|_X \leq 2\gamma_A R,$$

to assert that $A(t, u(t))X^m(t) \rightharpoonup A(t, u(t))u(t)$ in X and, hence, $A^m(t, X^m(t-h))X^m(t) \rightharpoonup A(t, u(t))u(t)$ in X, as $m \to \infty$. To show that $A(t, u(t))u(t)$ is weakly continuous on J_0, let $\{t_k\} \subset J_0$ be a sequence such that $t_k \to t$,as $k \to \infty$. Then, $u(t_k) \to u(t)$ in X as $k \to \infty$ and we follow the same arguments as above to prove the $A(t_k, u(t_k))u(t_k) \rightharpoonup A(t, u(t))u(t)$ in X as $k \to \infty$. For this, consider

$$\begin{aligned}
\|A(t_k, u(t_k))u(t_k) - A(t, u(t))u(t)\|_X &\leq \|(A(t_k, u(t_k)) - A(t, u(t)))u(t_k)\|_X \\
&\quad + \|A(t, u(t)(u(t_k) - u(t))\|_X
\end{aligned}$$

So

$$\begin{aligned}\|(A(t_k,u(t_k)) - A(t,u(t)))u(t_k)\|_X &\leq \|(A(t_k,u(t_k)) - A(t,u(t_k)))u(t_k)\|_X \\ &\quad + \|(A(t,u(t_k)) - A(t,u(t)))u(t_k)\|_X \\ &\leq L_A|t_k - t|\|u(t_k)\|_X \\ &\quad + \mu_A\|u(t_k) - u(t)\|_X(R + \|u_0\|_Y) \to 0\end{aligned}$$

as $k \to \infty$ since $t_k \to t$ and $u(t_k) \to u(t)$. We again use Lemma 1.3.31 and the fact that

$$\|A(t,u(t))(u(t_k) - u(t))\|_X \leq 2\gamma_A R,$$

to assert that $A(t,u(t))u(t_k) \rightharpoonup A(t,u(t))u(t_k)$ in X and, hence, $A(t_k,u(t_k))u(t_k) \rightharpoonup A(t,u(t))u(t)$ in X, as $k \to \infty$.

The Bochner integrability of $A(t,u(t))u(t)$ can be established in the similar way as in lemma 1.3.36. Now from (6.2.23), for each $x^* \in X^*$ we have

$$< U^m(t), x^* > = < u_0, x^* > + \int_0^t < -A^m(s, X^m(s-h))X^m(s) + f^m(s), x^* > ds.$$

Letting $m \to \infty$ using the bounded convergence theorem, we get

$$< u(t), x^* > = < u_0, x^* > + \int_0^t < -A(s,u(s))u(s) + f(s,u(s),G(u)(s)), x^* > ds.$$

The continuity of the integrand implies that $< u(t), x^* >$ is continuously differentiable on J_0. The Bochner integrability of $A(t,u(t))u(t)$ implies that the strong derivative of $u(t)$ exists a.e. on J_0 and

$$\frac{du}{dt}(t) + A(t,u(t))u(t) = f(t,u(t),G(u)(t)), \quad \text{a.e. on } J_0.$$

Since $u(0) = u_0$, u is a strong solution to (6.2.1).

Now, we establish the uniqueness and the continuous dependence on the initial data of a strong solution to (6.2.1).

Uniqueness. Let v be the another strong solution to (6.2.1) on J_0. Let $U = u - v$. Then, for a.e. $t \in J_0$

$$\begin{aligned}\left\langle \frac{dU}{dt}(t), F(U(t)) \right\rangle &+ < (\beta I + A(t, u(t)))U(t), F(U(t)) > \\ &= \beta \|U(t)\|_X^2 + < (A(t, u(t)) - A(t, v(t)))v(t), F(U(t)) > \\ &\quad + < f(t, u(t), G(u)(t)) - f(t, v(t), G(v)(t)), F(U(t)) > .\end{aligned}$$

Using the m-accretivity of $\beta I + A(t, w)$ and the assumptions on $A(t, w)$ for $(t, w) \in [0, T] \times W$ and $f(t, u, v)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_X^2 \leq C_R \|U\|_{C(J_t, X)}^2,$$

where $J_t = [0, t]$ and

$$C_R = \beta + \mu_A(R + \|u_0\|_Y) + \mu_f(R)(1 + \mu_G(R)).$$

Integrating the above inequality on $(0, t)$ and taking the supremum, we get

$$\frac{1}{2}\|U\|_{C(J_t, X)}^2 \leq C_R \int_0^t \|U\|_{C(J_s, X)}^2 ds.$$

Applying the Gronwall's inequality we get $U \equiv 0$ on J_0.

Continuous Dependence. Let $v_0 \in B_Y(u_0, R_0)$.Then

$$\|Sv_0 - z_0\|_X \leq \|Sv_0 - Su_0\|_X + \|Su_0 - z_0\|_X \leq 2R_0.$$

Hence

$$
\begin{aligned}
(1+e^{2\theta T})[\|Sv_0 - z_0\| + T_0\{\gamma_A\|z_0\|_Y + \gamma_P\|z_0\|_X + M\}] &\leq 3(1+e^{2\theta T})R_0 \\
&= R.
\end{aligned}
$$

We may proceed as before to prove the existence of $v_j^n \in W_R$ satisfying the scheme (6.2.1) with u_j^n and u_0 replaced by v_j^n and v_0, respectively. The convergence of v_j^n to $v(t)$ can be proved in a similar manner. Let $U = u - v$. Then, following the same steps used to prove the uniqueness, we get for a.e. $t \in J_0$,

$$
\frac{1}{2}\frac{d}{dt}\|U(t)\|_X^2 \leq C_R\|U\|_{C(J_t,X)}^2.
$$

Integrating the above inequality on $(0,t)$ and taking the supremum, we get

$$
\frac{1}{2}\|U\|_{C(J_t,X)}^2 \leq \frac{1}{2}\|U(0)\|_X^2 + C_R\int_0^t \|U\|_{C(J_s,X)}^2 ds.
$$

Applying the Gronwall's inequality, we get

$$
\|U\|_{C(J_t,X)} \leq C\|U(0)\|_X,
$$

where C is a positive constant. This proves the required result. This completes the proof of the theorem.

6.3 Regular Solution

In this section, we shall prove the existence of a unique local mild solution and a local classical solution to the quasi-linear integro-differential equation.

Let X and Y be two real Banach spaces, not necessarily reflexive as considered in the earlier section, such that the embedding $Y \hookrightarrow X$ is dense and continuous. Again, consider the following quasi-linear integro-differential equation in X:

$$\frac{du}{dt}(t) + A(t, u(t))u(t) = f(t, u(t), G(u)(t)), \quad 0 < t \leq T, \quad u(0) = u_0, \tag{6.3.1}$$

where $0 < T < \infty$, $A(t, u)$ is a linear operator in X, for each u in an open subset W of X, G is a nonlinear Volterra operator defined from $C(J, X)$ into $C(J, X)$, where $J = [0, T]$ and the nonlinear map f defined from $J \times W \times W$ into X. We follow the approach of T. Kato [56, 61, 63] to establish the existence of a unique *classical solution* to (6.3.1) under the assumptions (H1)-(H8) to be stated below.

In this section, first we mention some notations and preliminaries. Then, we establish the existence of a unique local mild solution using the contraction mapping theorem and the existence of a local classical solution to (6.3.1) under some additional conditions.

Let X and Y be as mentioned before and $\bar{B}_Z(r, z_0)$ be the closer of the open ball $B_Z(r, z_0) = \{z \in Z \mid \|z - z_0\|_Z < r\}$ with the radius r and the center at z_0 in a Banach space Z. The space of all bounded linear operators from X to Y is denoted by $B(X, Y)$ and $B(X, X)$ is written as $B(X)$. Let J denote the interval $[0, T]$. The space $C^m(J, Z)$ represents the space of all m-times continuously differentiable function defined on J into Z, $m = 1, 2, ...$; endowed with the supremum norm

$$\|u\|_{C^m(J,Z)} = \sum_{1 \leq i \leq m} \sup_{t \in J} \|u^{(i)}(t)\|, \quad u \in C^m(J, Z),$$

where $u^{(i)}$ denote the ith derivative of u with $u^{(0)} = u$. Let W be a subset of X such that for every $(t, w) \in J \times W$ $A(t, w)$ is a infinitesimal generators of C_0-semigroups $S_{t,w}(s)$, $s \geq 0$ on X. The family of operators $\{A(t, w)\}$, $(t, w) \in J \times W$, is called *stable*

if there exists constants $M \geq 1$ and ω, known as *stability constants*, such that

$$\rho(A(t,w)) \supset (\omega, \infty) \quad \text{for} \quad (t,w) \in J \times W,$$

where $\rho(A(t,w))$ is the resolvent set of $A(t,w)$ and

$$\left\| \prod_{j=1}^{k} R(\lambda : A(t_j, w_j)) \right\|_{B(X)} \leq M(\lambda - \omega)^{-k} \quad \text{for} \quad \lambda > \omega$$

and every finite sequence

$$0 \leq t_1 \leq t_2 \leq .... \leq t_k \leq T, \quad w_j \in W, \quad 1 \leq j \leq k.$$

For a linear operator S in X, by the part $\tilde{S}$ of S in a subspace Y of X, we mean a linear operator $\tilde{S}$ with domain $D(\tilde{S}) = \{x \in D(S) \cap Y \ : \ Sx \in Y\}$ and values $\tilde{S}x = Sx$ for $x \in D(\tilde{S})$.

Let $S_{t,w}(s)$, $s \geq 0$, be the C_0-semigroup generated by $A(t,w)$, $(t,w) \in J \times W$. A subset Y of X called $A(t,w)$-admissible if Y is an invariant subspace of the operator $S_{t,w}(s)$, $s \geq 0$, and the restriction of $S_{t,w}(s)$ to Y is a C_0-semigroup in Y.

For the details of the above mentioned notions, the reader may refer to the chapters 5 and 6 in Pazy [77]. On the family of operators $\{A(t,w),(t,w) \in J \times W\}$, we make the same assumptions $(\tilde{H}1)$-$(\tilde{H}4)$ considered in §6.6.4 in Pazy [77] for a homogeneous quasi-linear evolution equation. For completeness, we restate those assumptions.

(H1) The family $\{A(t,w)\}$, $(t,w) \in J \times W$ is stable.

(H2) Y is $A(t,w)$-admissible for $(t,w) \in J \times W$ and the family $\{\tilde{A}(t,w),(t,w) \in J \times W\}$ of the parts $\tilde{A}(t,w)$ of $A(t,w)$ in Y, is stable in Y.

(H3) For $(t,w) \in J \times W$, $D(A(t,w)) \supset Y$, $A(t,w)$ is a bounded linear operator from Y to X and the map $t \mapsto A(t,w)$ is continuous in $B(Y,X)$ norm $\|\cdot\|_{Y \to X}$ for every $w \in W$.

(H4) There is a positive constant L_A such that

$$\|A(t,w_1) - A(t,w_2)\|_{Y\to X} \leq L_A\|w_1 - w_2\|,$$

for every $w_1,\ w_2 \in W$ and $0 \leq t \leq T$.

If $u \in C(J,X)$ is W-valued and The family $\{A(t,w),(t,w) \in J \times W\}$ of the operators satisfies (H1)-(H4), then there exists a unique evolution system $U_u(t,s)$, $0 \leq s \leq t \leq T$, in X satisfying

$$\text{(i)} \quad \|U_u(t,s)\| \leq Me^{\omega(t-s)}, \tag{6.3.2}$$

for $0 \leq s \leq t \leq T$, where M and ω are the stability constants;

$$\text{(ii)} \quad \frac{\partial^+}{\partial t}U_u(t,s)w|_{t=s} = A(s,u(s))w, \tag{6.3.3}$$

for $w \in Y$, $0 \leq s \leq T$;

$$\text{(iii)} \quad \frac{\partial}{\partial s}U_u(t,s)w = -U_u(t,s)A(s,u(s))w, \tag{6.3.4}$$

for $\omega \in Y$, $0 \leq s \leq T$.

Further, there exists a positive constant C_0 such that for every $u, v \in C(J,X)$ with values in W and every $y \in Y$, we have

$$\|U_u(t,s)y - U_v(t,s)y\|_X \leq C_0\|y\|_Y \int_s^t \|u(\tau) - v(\tau)\|_X d\tau. \tag{6.3.5}$$

For details of the above mentioned results, reader may refer to the Theorem 6.4.3 and Lemma 6.4.4 in Pazy [77].

We further assume :

(H5) for every $u \in C(J, X)$ satisfying $u(t) \in W$ for $t \in J$, we have

$$U_u(t,s)Y \subset Y, \quad t, s \in J, \quad s \le t$$

and $U_u(t,s)$ is strongly continuous in Y for s, $t \in J$, $s \le t$.

(H6) The closed convex subsets of Y are also closed in X.

(H7) The nonlinear map $G : C(J,X) \to C(J,X)$ satisfy the following.

(a) For all u, $v \in \bar{B}_{C(J,X)}(\tilde{u_0}, r)$,

$$\|G(u) - G(v)\|_{C(J,X)} \le \mu_G(r)\|u - v\|_{C(J,X)},$$

where $\mu_G(r)$ is a nonnegative and nondecreasing function and $\tilde{u_0} \in C(J,X)$ is defined by $\tilde{u_0} = u_0$ for all $t \in J$.

(b) The subspace $C(J,Y)$ of the space $C(J,X)$ is an invariant subspace of the map G, i.e. $G : C(J,Y) \to C(J,Y)$ and satisfies

$$\|G(u)(t)\|_Y \le \lambda_G(r) \quad \text{for} \quad u \in \bar{B}_Y(u_0, r),$$

where $\lambda_G(r)$ is a nonnegative and nondecreasing function.

(H8) The nonlinear map $f : J \times W \times W \to X$ satisfies.

(a) For $(t,u,v) \in J \times (W \cap Y) \times (W \cap Y)$, $f(t,u,v) \in Y$ and

$$\|f(t,u,v)\|_Y \le \lambda_f(r),$$

for all $(t,u,v) \in J \times W \times W$ with $\|u\|_Y + \|v\|_Y \le r$, where $\lambda_f(r)$ is a nonnegative and nondecreasing function.

(b) In both $Z = X$, Y, the nonlinear map f satisfies the Lipschitz-like condition

$$\|f(t_1, u_1, v_1) - f(t_2, u_2, v_2)\|_Z \leq \mu_f(r)[|\phi(t_1) - \phi(t_2)| + \|u_1 - u_2\|_Z + \|v_1 - v_2\|_Z],$$

for all t_1, $t_2 \in [0, T]$ and all u_i, $v_i \in \bar{B}_Y(u_0, r)$, $i = 1, 2$, where ϕ is a real-valued continuous function of bounded variation on $[0, T]$ and $\mu_f(r)$ is a nonnegative and nondecreasing function.

By a *mild solution* to (6.3.1) on $J = [0, T]$, we mean a function $u \in C(J, X)$ with values in W satisfying the integral equation

$$u(t) = U_u(t, 0)u_0 + \int_0^t U_u(t, s)f(s, u(s), G(u)(s))ds, \quad t \in J. \tag{6.3.6}$$

By the *Classical solution* u to (6.3.1) on $J = [0, T]$, we mean a function $u \in C(J, X)$ such that $u(t) \in Y \cap W$ for $t \in (0, T]$, $u \in C^1((0, T], X)$ and satisfies (6.3.1) in X. If there exists a $0 < T' \leq T$ and a function $u \in C(J', X)$, $J' = [0, T']$, such that u is a mild (classical) solution to (6.3.1) on J', then u is called a *local mild (classical)* solution to (6.3.1).

Now, we state and prove the following main result.

Theorem 6.3.1 *Suppose that $u_0 \in Y$ and the family $\{A(t, w)\}$ of linear operators for $t \in J = [0, T]$ and $w \in W = \{y \in Y : \|y - u_0\|_Y \leq r\}$, for fixed $r > 0$, satisfy the assumptions (H1)-(H6) and $A(t, w)u_0 \in Y$ with*

$$\|A(t, w)u_0\|_Y \leq C_A, \tag{6.3.7}$$

for all $(t, w) \in J \times W$.

Further, Suppose that nonlinear map G and f satisfy (H7) and (H8), respectively. Then there exists a unique local classical solution to (6.3.1).

Proof. First, we establish the existence of a unique local mild solution to (6.3.1). We note that from the assumption (H5), it follows that

$$\|U_u(t,s)\|_{B(Y)} \leq C_1, \tag{6.3.8}$$

for $s \leq t$, $s, t \in J$ and every $u \in C(J, X)$ with values in W. We choose

$$T_0 = \min\left\{T, \frac{r}{2C_A C_1}, \frac{r}{2C_1\lambda_f(R_1)}, \frac{1}{2P}\right\}, \tag{6.3.9}$$

where

$$P = C_0\|u_0\|_Y + Me^{\omega T}\mu_f(R_1)(1+\mu_G(r)) + C_0\lambda_f(R_1)T$$

and

$$R_1 = r + \|u_0\|_Y + \lambda_G(r).$$

Let S be the subset of $C(J_0, X)$ defined by

$$S = \{u \,:\, u \in C(J_0, X),\ u(0) = u_0,\ u(t) \in W \text{ for } t \in J_0\},$$

where $J_0 = [0, T_0]$. From (H6), it follows that S is a closed convex subset of $C(J_0, X)$. Next, we define a mapping $F : S \to S$ given by

$$Fu(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)f(s,u(s),G(u)(s))ds \tag{6.3.10}$$

and check that F is well defined. Clearly, $Fu(0) = u_0$, $Fu \in C(J_0, X)$ and (H5) implies that $Fu(t) \in Y$ for $t \in J_0$ and now, it remains to show that $\|Fu(t) - u_0\|_Y \leq r$ for $t \in J_0$. Consider

$$Fu(t) - u_0 = U_u(t,0)u_0 - u_0 + \int_0^t U_u(t,s)f(s,u(s),G(u)(s))ds. \tag{6.3.11}$$

Integrating (6.3.4) in X from 0 to t, we find

$$U_u(t,0)u_0 - u_0 = \int_0^t U_u(t,\tau)A(\tau,u(\tau))u_0 d\tau. \tag{6.3.12}$$

Hence, using (6.3.7) and (6.3.8) in (6.3.12), we have

$$\|U_u(t,0)u_0 - u_0\|_Y \;\leq\; C_1 C_A T_0 \;\leq\; \frac{r}{2}. \tag{6.3.13}$$

Also, using (6.3.8) and (H8), we have

$$\left\| \int_0^t U_u(t,s) f(s,u(s),G(u)(s)) ds \right\|_Y \;\leq\; C_1 \lambda_f(R_1) T_0 \;\leq\; \frac{r}{2}, \tag{6.3.14}$$

since $\|u(s)\|_Y + \|G(u)(s)\|_Y \leq R_1$. Using (6.3.13) and (6.3.14) in (6.3.11), we see that F is well defined. Now, suppose that $u, v \in S$, then

$$\begin{aligned} Fu(t) - Fv(t) &= (U_u(t,0) - U_v(t,0))u_0 \\ &\quad + \int_0^t [U_u(t,s) f(s,u(s),G(u)(s)) \\ &\qquad - U_v(t,s) f(s,v(s),G(v)(s))] ds \\ &= T_1 + T_2, \end{aligned} \tag{6.3.15}$$

where T_1 and T_2 represent the first and second terms on the right hand side of (6.3.15). We use (6.3.5) to obtain

$$\|T_1\|_X \;\leq\; C_0 \|u_0\|_Y T_0 \|u - v\|_{C(J_0,X)}.$$

Now, we use (H7), (H8) and (6.3.5) and get

$$\begin{aligned} \|T_2\|_X &\leq \left\| \int_0^t U_u(t,s)[f(s,u(s),G(u)(s)) - f(s,v(s),G(v)(s))] ds \right\|_X \\ &\quad + \left\| \int_0^t [U_u(t,s) - U_v(t,s)] f(s,v(s),G(v)(s)) ds \right\|_X \\ &\leq [M e^{\omega T} \mu_f(R_1)(1 + \mu_G(r)) + C_0 \lambda_f(R_1) T] T_0 \|u - v\|_{C(J_0,X)}, \end{aligned}$$

since, we have

$$\begin{aligned}
\|f(s,u(s),G(u)(s)) &- f(s,v,(s),G(v)(s)\|_X \\
&\le \mu_f(R_1)[\|u(s)-v(s)\|_X + \|G(v)(s)-G(u)(s)\|_X] \\
&\le \mu_f(R_1)[\|u-v\|_{C(J_0,X)} + \|G(u)-G(v)\|_{C(J_0,X)}] \\
&\le \mu_f(R_1)(1+\mu_G(r))\|u-v\|_{C(J_0,X)}.
\end{aligned}$$

Hence, from (6.3.15), we have

$$\begin{aligned}
\|Fu-Fv\|_{C(J_0,X)} &\le PT_0\|u-v\|_{C(J_0,X)} \\
&\le \frac{1}{2}\|u-v\|_{C(J_0,X)}.
\end{aligned}$$

Thus, F is a contraction map from S to S. Since S is closed in X, by the contraction mapping theorem, F has a unique fixed point $u \in S$ which is the local mild solution to (6.3.1).

Now, we show that $u \in C(J_0,Y)$. For $s,t \in J_0$ with $s \le t$, we have

$$\begin{aligned}
u(t)-u(s) &= (U_u(t,0)-U_u(s,0))u_0 \\
&\quad + \int_0^s (U_u(t,\eta)-U_u(s,\eta))f(\eta,u(\eta),G(u)(\eta))d\eta \\
&\quad + \int_s^t U_u(t,\eta)f(\eta,u(\eta),G(u)(\eta))d\eta.
\end{aligned}$$

Since $U_u(t,s)$ is strongly continuous in Y for $s,t \in J$, $s \le t$ so, for every $\epsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$t_1,t_2 \in J_0 \quad \text{with} \quad |t_1-t_2| \le \delta_1 \quad \Rightarrow \quad \|U_u(t,0)-U_u(s,0)\|_{B(Y)} \le \frac{\epsilon}{3\|u_0\|_Y},$$

$$t_1,t_2 \in J_0 \quad \text{with} \quad |t_1-t_2| \le \delta_2 \quad \Rightarrow \quad \|U_u(t,\eta)-U_u(s,\eta)\|_{B(Y)} \le \frac{\epsilon}{3\lambda_f(R_1)T_0}.$$

Let

$$\delta = \min\{\delta_1, \delta_2, \frac{\epsilon}{3C_1\lambda_f(R_1)}\}.$$

Then, for $s, t \in J_0$

$$|t-s| \leq \delta \Rightarrow \|u(t) - u(s)\|_Y \leq \epsilon.$$

Thus, $u \in C(J_0, Y)$.

Consider the following linear evolution equation

$$\begin{aligned} \frac{dv(t)}{dt} + B(t)v(t) &= h(t), \quad 0 < t \leq T_0, \\ v(0) &= u_0, \end{aligned} \tag{6.3.16}$$

where $B(t) = A(t, u(t))$ and $h(t) = f(t, u(t), G(u)(t))$ for $t \in J_0$ and u is the unique fixed point of F in S. We note that $B(t)$ for $t \in J$ be the infinitesimal generator of a C_0 semigroup $S_t(s)$, $s \geq 0$ on X and satisfies (H1)-(H3) of Theorem 1.3.23.

We have to prove that $h \in C(J_0, Y)$. For s, $t \in J_0$ (we assume with out loss of generality that $s \leq t$), we have

$$\begin{aligned} \|h(t) - h(s)\|_Y &= \|f(t, u(t), G(u)(t)) - f(s, u(s), G(u)(s))\|_Y \\ &\leq \mu_f(R_1)[|\phi(t) - \phi(s)| + \|u(t) - u(s)\|_Y \\ &\quad + \|G(u)(t) - G(u)(s)\|_Y]. \end{aligned}$$

As ϕ is a continuous function of bounded variation on J, $u \in C(J_0, Y)$ and $G(u) \in C(J_0, Y)$ for $u \in C(J_0, Y)$. So, for every $\epsilon > 0$, there exist $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$t_1, t_2 \in J_0 \quad \text{with} \quad |t_1 - t_2| \leq \delta_1 \quad \Rightarrow \quad |\phi(t) - \phi(s)| \leq \frac{\epsilon}{3\mu_f(R_1)},$$

$$t_1, t_2 \in J_0 \quad \text{with} \quad |t_1 - t_2| \leq \delta_2 \quad \Rightarrow \quad \|u(t) - u(s)\|_Y \leq \frac{\epsilon}{3\mu_f(R_1)},$$

$$t_1, t_2 \in J_0 \quad \text{with} \quad |t_1 - t_2| \leq \delta_3 \quad \Rightarrow \quad \|G(u)(t) - G(u)(s)\|_Y \leq \frac{\epsilon}{3\mu_f(R_1)}.$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, for $s, t \in J_0$, we have

$$|t - s| \leq \delta \quad \Rightarrow \quad \|h(t) - h(s)\|_Y \leq \epsilon$$

Thus, $h \in C(J_0, Y)$. Theorem 1.3.24 implies that there exists a unique function $v \in C(J_0, Y)$ such that $v \in C^1(J_0/\{0\}, X)$ satisfying (6.3.16) in X and v is given by

$$v(t) = U_u(t,0)u_0 + \int_0^t U_u(t,s)f(s,u(s),G(u)(s))ds, \quad t \in J_0,$$

where $U_u(t,s)$, $0 \leq s \leq t \leq T_0$ is the evolution system generated by the family $\{A(t,u(t))\}$, $t \in J_0$, of linear operator s in X. The uniqueness of v implies that $v \equiv u$ on J_0 and hence, u is a unique local classical solution to (6.3.1). This completes the proof of the theorem.

Chapter 7

Scope for Further Research

The outcome of any serious research can only be to make two questions grow where only one grew before.

— Thorstein Veblen

In Chapter 2 we have proved the convergence results for the approximate solution to a first order Sobolev type evolution equation under more general conditions than the ones considered by [39]. In order to show the continuity of the map $t \mapsto (S_n u)(t)$, we assume Hölder continuity in t on $A^\beta g(t,u)$ only and not on the function $f(t,u)$ as assumed in [39]. After proving that S_n is a strict contraction, we conclude that the solution $u_n(t)$ of approximate integral equation is Hölder continuous which in turn implies the Hölder continuity of the solution $u(t)$.

To get the approximate solution of this evolution equation we have to solve a system of first order ordinary differential equations satisfied by the coefficients $\alpha_i^n(t)$. Finding the solution of this system would not be so straightforward since derivatives of α_i^n are not given explicitly. It would depend on the forms of g and f. We may solve this system by using numerical techniques for solving ordinary differential equations. For

further research work we plan to consider some initial boundary value problems which may be reformulated as Sobolev type evolution equations considered in the present work. We would like to study the behavior of solution obtained numerically with the help of corresponding systems of first order ordinary differential equations.

Neutral Functional Differential Equations

Consider the boundary value problem

$$\frac{\partial}{\partial t}\left[u(t,x)+\int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,\eta,x)u(s,\eta)d\eta ds\right] = \frac{\partial^2}{\partial x^2}u(t,x)+a_0(x)u(t,x),$$
$$+\int_{-\infty}^{t} a(s-t)u(s,x)ds + a_1(t,x), \quad t\geq 0, \quad 0\leq x\leq \pi,$$
$$u(t,0)=u(t,\pi)=0, \quad t\geq 0,$$
$$u(\theta,x)=\phi(\theta,x), \quad \theta\leq 0, \quad 0\leq x\leq \pi,$$

where the functions a_0, a, a_1, b_1 and ϕ satisfy appropriate conditions. We can reformulate this problem in the form of an abstract neutral functional differential equation with unbounded delay of the form

$$\begin{aligned} \frac{d}{dt}(x(t)+G(t,x_t)) &= Ax(t)+F(t,x_t), \quad t\geq\sigma, \\ x_\sigma = \varphi \in \Omega, & \end{aligned} \tag{7.0.1}$$

in a Banach space X where Ω is an open subset of some abstract phase space $\mathcal{B}$. The history function $x_t : (\infty, 0] \to X$ given by $x_t(\theta) := x(t+\theta)$ belongs to $\mathcal{B}$, and the nonlinear functions F, $G : [0,a]\times\Omega \to X$ are continuous and A generates an analytic semigroup $T(t)$ of bounded linear operators on X and $0\leq\sigma<a$.

A mild solution for (7.0.1) is a function $x : (-\infty, \sigma+b) \to X$, $b>0$, such that $x_\sigma = \varphi$, the restriction of $x(.)$ on the interval $[\sigma, \sigma+b)$ is continuous and for each

$\sigma \le t < \sigma + b$ the function $AT(t-s)F(s, x_s)$, $s \in [\sigma, t)$, is integrable and satisfies the integral equation

$$\begin{aligned} x(t) &= T(t-\sigma)[\varphi(0) + F(\sigma, \varphi)] - F(t, x_t) \\ &\quad - \int_\sigma^t AT(t-s)F(s, x_s)ds + \int_\sigma^t T(t-s)G(s, x_t)ds, \quad \sigma \le t. \end{aligned}$$

We plan to establish similar approximation results for the equation (7.0.1) as we have done for the Sobolev type case. Further, we want to obtain error estimates in the Faedo-Galerkin procedure for the evolution equations considered in the present study.

In Chapter 3 and Chapter 5, we have proved the convergence of the approximate solution to the second order semi-linear evolution and integro-differential equations. In these cases, we get a pair of systems of first order ordinary integro-differential equations which can not be easily solved. How to solve these systems will depend on the form of nonlinear functions f and g. If f and g depend on unknown function only and not on the time derivative of the unknown function, then we do not require to solve the second system. We may get the solution by solving the first system only which involves α_i^n. We have planed to develop a method to solve these type of systems.

In Chapter 6, we have proved the existence of a unique strong solution which continuously depends on the initial data with the help of the method of semi-discretization in time. Using this method we would like to prove the existence result for the second order semi-linear integro-differential equation considered in Chapter 5. We plan to consider the following discretization scheme.

$$\delta^2 u_j^n + A\delta u_j^n + Bu_{j-1}^n = f_j^n + h\sum_{i=0}^{j-1} k_{ji}^n g_i^n$$

where

$$\delta^2 u_j^n = \frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{h^2}, \quad j = 1, 2 \cdots n;$$

$$\delta u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \quad j = 0, 1, \cdots n;$$

$$f_j^n = f(t_j^n, u_{j-1}^n, \delta u_{j-1}^n), \quad j = 1, 2 \cdots n;$$

$$k_{ji}^n = k(t_j^n - t_i^n), \quad 1 \leq i \leq j \leq n;$$

$$g_j^n = g(t_j^n, u_{j-1}^n, \delta u_{j-1}^n), \quad j = 1, 2 \cdots n.$$

We would like to obtain the error estimates also, i.e., the problem of estimating the difference between the actual value of the solution u and its approximation u_n obtained at the point of discretization by Rothe's method which is important from the numerical point of view.

Second Order Nonlinear Differential Equations

Let X and Y be two real Hilbert spaces such that $Y \subset X$ and let the inclusion mapping of Y into X be continuous and densely defined. Let Y' denote the dual of Y. We identify X with its own dual X'. Consider the problem

$$\begin{aligned} &\frac{d^2u}{dt^2}(t) + A(\frac{du}{dt})(t) + Bu(t) \ni f(t, u(t), \frac{du}{dt}(t)), \quad \text{a.e. on} \quad (0,T), \\ &u(0) = u_0, \quad \frac{du}{dt}(0) = v_0, \end{aligned} \tag{7.0.2}$$

where A is a maximal monotone set in $Y \times Y'$ and $A \in L(Y, Y')$ is assumed to satisfy the following coercivity condition

$$(Au, u) + \alpha|u|^2 \;\geq\; \omega\|u\|^2, \quad \text{for every} \quad Y,$$

where $\alpha \in R^1$ and $\omega > 0$. The nonlinear map f is defined from $(0,T) \times Y \times Y$ to H.

In chapter 3 we have established the convergence results for the approximate solutions of a second order semi-linear evolution equation. Our aim is to extend these results for the second order nonlinear differential inclusion (7.0.2).

List of Papers Accepted and Communicated

1. D. Bahuguna, S. Singh and **R. Shukla**, 'Application of Method of Semidiscretization in Time to Semilinear Viscoelastic Systems', *Differential Equations and Dynamical Systems*, to appear.

2. D. Bahuguna and **R. Shukla**, 'Method of semidiscretization in time to quasilinear integrodifferential equations', *Nonlinear Dynamics and Systems Theory*, to appear.

3. D. Bahuguna and **R. Shukla**, 'Approximations of solutions to nonlinear Sobolev type evolution equations', *Electronic J. Diffenential Equations*, to appear.

4. D. Bahuguna and **R. Shukla**, 'Approximations of solutions to second order semilinear evolution equations', Communicated.

5. D. Bahuguna and **R. Shukla**, 'Application of Rothe Method to Abstract Quasilinear Implicit Integrodifferential Equations in Reflexive Banach Space', Communicated.

6. D. Bahuguna and **R. Shukla**, 'Approximations of Solutions to Second Order Semilinear Integrodifferential Equations', Communicated.

7. **R. Shukla** and D. Bahuguna, 'Abstract Second Order Semilinear Integrodifferential Equation in Banach Space', Communicated.

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